

Lower bound limit analysis of unsaturated soil slope stability under fluctuating water levels using finite elements and convex optimization

Jianfeng Zhou^{a,1}, Qi Zhang^{b,*}, Haoliang Zhang^b, Xudong Kang^c, Zilong Zhang^b

^a College of Civil Engineering, Huaqiao University, Xiamen, 361021, China

^b Department of Civil and Environmental Engineering, Faculty of Science and Technology, University of Macau, Taipa, 999078, Macao Special Administrative Region of China

^c Department of Civil and Environmental Engineering, The Hong Kong Polytechnic University, Hung Hom, Kowloon, 999077, Hong Kong Special Administrative Region of China

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ABSTRACT

Extensive evidence indicates that slope failure is frequently associated with intensive rainfall, but the coupled effects of matric suction and fluctuating water levels under such events have not been sufficiently investigated. This work develops a lower bound limit analysis framework, based on finite elements and convex optimization, to assess the stability of unsaturated slopes under fluctuating water levels. Within Bishop's effective stress framework, a novel convex formulation of the load multiplier λ_F considering the mixed-form Richards equation and finite element discretization is proposed. A rigorous lower bound solution for the Factor of Safety (FoS) of an unsaturated slope is derived by coupling the strength reduction technique with CVX, a MATLAB package for disciplined convex programming. Through model validation, the proposed convex formulation greatly outperforms the original Linear Program (LP) formulation, and the performance of CVX is highly competitive compared with other specific solvers in the literature. The model application section systematically summarizes the relative magnitudes and variation patterns of FoS under four typical slope seepage scenarios: slow reservoir drawdown, non-trivial steady-state reservoir drawdown considering external water supply, rapid transient reservoir drawdown, and post-drawdown rainfall. Results indicate that neglecting matric suction leads to an underestimation of slope stability, and when matric suction is incorporated, the $\chi = S_e$ model yields a more conservative estimate of the FoS compared to other χ models. During reservoir drawdown, the asynchronous response between the internal and external phreatic surfaces significantly influences the FoS. Consequently, two bounding scenarios, i.e., slow drawdown FoS and the initially sharply declining FoS during rapid drawdown, can be used to establish a reasonable prediction range for the FoS in practice. These results provide valuable guidance for the safety assessment of slopes subjected to fluctuating water levels, as well as for reservoir management.

1. Introduction

Rainfall infiltration and reservoir water-level fluctuations drive spatiotemporal variation of soil saturation in natural and engineered slopes. For example, the bank slopes in the Three Gorges Reservoir area are frequently subjected to rapid and significant water-level fluctuations over short periods (Sun et al., 2017). In addition, rainfall-after-drawdown situations frequently occur in practical reservoir operations. Some reservoirs in Florida (United States) and Taiwan (Republic of China) are deliberately drawn down prior to the onset of the rainy season to secure sufficient flood-control storage (Özer and Bromwell, 2012;

Chou and Wu, 2013). The consequent dynamics in matric suction directly affects shear strength of soils and thereby slope stability (Griffiths and Lu, 2005; Huang and Jia, 2009). Numerous engineering incidents indicate that rapid water level fluctuations and prolonged rainfall are the primary triggers of landslides (Iverson, 2000; Wu et al., 2020). Therefore, a robust evaluation of unsaturated slope stability under these complex scenarios by incorporating hydro-mechanical coupling effects is imperative for the mitigation and prevention of such geohazards.

Among mainstream slope stability assessment methods, the limit equilibrium method (LEM) is widely used for its simplicity, and it

* Corresponding author.

E-mail address: qzhang@um.edu.mo (Q. Zhang).

¹ These authors contributed equally to this work.

has also been extended to the bearing capacity problem with unsaturated soils and combined loading (Afsharpour et al., 2022; Vahedifard et al., 2016). However, the LEM's reliance on prescribed assumptions regarding the stress field and failure mechanism often impedes its ability to systematically address complex boundary conditions and multiphysics coupling. Conversely, the limit analysis method (LAM), based on plasticity theorems, provides upper and lower bound solutions that yield strictly unsafe and safe-side estimates, respectively (Huang et al., 2025, 2026; Zhang et al., 2024). The lower bound approach, in particular, identifies a stress field that satisfies static equilibrium and the yield criterion, ensuring that the computed ultimate load (or safety factor) does not exceed the true value, thereby offering safe-side guidance for engineering practice (Chen and Scawthorn, 1970). In addition, the finite element method (FEM) has gained prominence due to its adaptability to diverse constitutive models and its capacity to integrate seepage, dynamic loading, and other physical fields (Huang and Jia, 2009; Vicente Da Silva, 2025; Xiao et al., 2024). Nevertheless, FEM solutions do not inherently provide mathematically rigorous upper or lower bounds, and the safety factor is sensitive to mesh generation, convergence criteria, load stepping, and strength reduction paths. To bridge these gaps, the lower bound finite element (LBFE) method has been proposed that integrates the analysis of the lower bound limit with FEM (Sloan, 1988), further enhancing the ability to handle complex geometries, materials, and boundary conditions. The LBFE framework has shown strong applicability in classic problems such as slope stability, tunnel stability, bearing capacity of foundations, and retaining wall stability (Tschuchnigg et al., 2015b,a; Foroutan Kalourazi et al., 2020; Payan et al., 2022; Sahu et al., 2025; Keshavarz et al., 2025; Fathipour et al., 2023a; Bahmani Tajani et al., 2023).

Most existing lower bound finite element (LBFE) analyses of slope stability have been conducted under the assumption of fully saturated conditions. Li et al. (2010) employed the LBFE method to evaluate the stability of three-dimensional slopes in purely cohesive and c - ϕ soils. Lim et al. (2015) extended this approach to layered soils, using it to assess the stability of slopes on two-layer undrained clays. Qian et al. (2017) presented LBFE solutions for rock slope safety factors based on the Hoek–Brown failure criterion. Zhou et al. (2023) combined the pseudo-static method with the LBFE method to evaluate seismic slope stability. More recently, Vicente Da Silva (2025) incorporated a nonlinear Mohr–Coulomb criterion into the finite element limit analysis framework and discussed its applicability to slope stability. Bozorgpour et al. (2025) integrated random field theory with the LBFE method to assess slope stability in heterogeneous and anisotropic soils. However, the majority of natural slopes exist in an unsaturated state (Griffiths and Lu, 2005), and numerous field or laboratory tests have demonstrated that the stability of unsaturated slopes is significantly influenced by the matric suction (Chen et al., 2018; Tao et al., 2024). Some studies have employed the LBFE method to investigate the stability of retaining walls or foundations on unsaturated soils under different hydraulic conditions, indicating that seepage has a significant impact on the stability of geotechnical structures (Fathipour et al., 2020, 2022, 2023b). For example, Fathipour et al. (2020) investigated the active and passive earth pressures acting on retaining structures with unsaturated backfills using an LBFE framework coupled with second-order cone programming, showing that suction stress reduces active pressure while increasing passive pressure.

Significant efforts have been directed toward assessing unsaturated slope stability via conventional FEM or LAM. Lane and Griffiths (2000) used FEM to assess slope stability during reservoir drawdown, without considering matric suction effects. Huang and Jia (2009) applied strength reduction FEM to evaluate slope stability under transient seepage and found that the safety factor increases with time and eventually stabilizes following rapid drawdown. To account for inherent uncertainties, Le et al. (2015) used random FEM to estimate the failure probability of slopes with spatially variable void ratios,

while Johari and Talebi (2019) evaluated failure probability by incorporating spatially variable hydraulic conductivity. Within the limit analysis framework, Zhang and Yang (2021) employed an upper bound approach to assess the seismic stability of unsaturated slopes, demonstrating that when the effective internal friction angle and slope angle are 25° and 45° , respectively, increasing matric suction from 0 to 120 kPa raises the safety factor by approximately 130%. However, regardless of the method used to analyze unsaturated slope stability, the essential ingredients are the effective stress framework and the constitutive relationship or strength criterion. As reviewed by Sheng et al. (2008), the two most widely used effective stress frameworks are Bishop's effective stress framework and the suction-stress concept (dual-stress variable framework). Corresponding constitutive models include purely mechanical models based on Bishop's effective stress, as well as fully coupled hydro-mechanical models such as the Barcelona Basic Model (Alonso et al., 1990). Similarly, the strength criteria include the Mohr–Coulomb criterion using Bishop's effective stress, and the ϕ^b approach proposed by Fredlund et al. (1978). The recent effective stress framework of unsaturated soils proposed by Prof. Ning Lu (Lu et al., 2010) is also well-recognized. Therefore, before conducting an unsaturated slope stability analysis, the mechanical model to be used must be determined in advance.

Despite these advances, under the LBFE framework, a systematic study of unsaturated soil slope stability under four typical slope seepage scenarios, namely slow reservoir drawdown, steady-state reservoir drawdown considering external water supply, rapid reservoir drawdown, and post-drawdown rainfall, remains absent. Although Zhou and Qin (2022) has attempted to address this issue, they failed to clearly distinguish the above four scenarios. Additionally, their work is performed within the upper bound finite element framework, while the lower bound solution has not been addressed. As a result, the results of different scenarios were mixed together, hindering a clear understanding. This study aims to address this knowledge gap by proposing an LBFE framework for a comprehensive evaluation of unsaturated slope stability, considering water-level fluctuations and seepage regime transitions. The remaining contents are organized as follows. In Section 2, the framework is developed by first establishing finite element formulations for both transient and steady-state seepage. Subsequently, a convex optimization problem is formulated using the disciplined convex programming (DCP) ruleset and solved using CVX, a package for specifying and solving convex problems. Section 3 focuses on the overall procedure and important considerations such as the boundary condition, potential seepage face review, and strength reduction technique. The proposed approach is validated through comparisons with established literature in Section 4, confirming its advantage and competitiveness. Finally, a systematic study of unsaturated soil slope stability under four typical slope seepage scenarios is provided in Section 5. The limitations and conclusions are given in the end.

2. Methodology

This section begins with a concise review of the finite element formulation of the unsaturated flow equation (i.e., Richards equation), followed by an exposition of the finite element lower bound formulation. It is worth emphasizing that although both formulations share the same word “finite element”, these two formulations are fundamentally different. The former is the standard pressure/head-based finite element method for seepage problems, whereas the latter constitutes a specialized construct exclusive to limit analysis.

In the lower bound approach, the stress field is interpolated directly, using neither a weak form nor a matrix form (e.g., $Kd = F$). Additionally, in standard finite element analysis, different elements share the same node. However, in the finite element lower bound formulation, as described by Fig. 9 in Sloan (2013), each node is assigned exclusively to a single element carrying distinct stress variables, even when multiple nodes coincide geometrically. Consequently, for a triangular element mesh, the total number of nodes n_{node} in the lower bound limit analysis is equal to $3 \times n_{\text{elm}}$, where n_{elm} is the total number of triangular elements.

2.1. Finite element unsaturated flow formulation

For unsaturated water flow through porous media, let θ denote the volumetric water content (the ratio of volume occupied by water to the total mixture volume), ρ_w the water density (constant), $\mathbf{q}_w$ the Darcy seepage velocity, and $\mathfrak{S}$ the mass-specific source/sink term (typically root water uptake). Applying mass conservation to a spatially-fixed control volume Ω^0 yields the following integral balance

$$\frac{\partial}{\partial t} \int_{\Omega^0} \rho_w \theta dV = - \int_{\partial\Omega^0} \rho_w \mathbf{q}_w \cdot \mathbf{n} dA + \int_{\Omega^0} \rho_w \mathfrak{S} dV, \quad (1)$$

where $\partial\Omega^0$ denotes the boundary of Ω^0 , and $\mathbf{n}$ is the outward-pointing unit normal vector defined on $\partial\Omega^0$. Next, using the divergence theorem and recognizing that Eq. (1) holds for any arbitrary Ω^0 , the integral signs may be dropped to obtain the localized mass balance equation

$$\frac{\partial \theta}{\partial t} + \nabla \cdot \mathbf{q}_w = \mathfrak{S}, \quad (2)$$

which is exactly the famous Richards equation (Richards, 1931). In the numerical examples presented in this study, the source/sink term is set to zero ($\mathfrak{S} = 0$) for simplicity. However, it should be emphasized that the proposed approach is fully applicable to general cases with a non-zero $\mathfrak{S}$. The primary variable in Eq. (2) is the pore water pressure p_w . In unsaturated soil mechanics, p_w can attain negative values within the unsaturated zone (Song and Borja, 2014), and it is preferred to use the hydraulic pressure head h as the primary variable, which could be converted from p_w using $h = p_w/\gamma_w$, where $\gamma_w = \rho_w g$ denotes the unit weight of water.

Both θ and $\mathbf{q}_w$ would depend on h , and the functional relationship between θ and h is commonly known as the soil–water characteristic curve (SWCC), or water retention curve (Song and Borja, 2014). In this work, the van Genuchten SWCC model (Van Genuchten, 1980) is adopted, defined as

$$\theta = \theta_r + \frac{\theta_s - \theta_r}{\left[1 + (\alpha_{vG} \langle -h \rangle)^{n_{vG}}\right]^{1 - \frac{1}{n_{vG}}}}, \quad S_e = \frac{\theta - \theta_r}{\theta_s - \theta_r}, \quad (3)$$

where θ_s and θ_r denote the saturated and residual water contents, respectively, with θ_s typically equal to the soil porosity. The term S_e represents the effective saturation, which is strictly bounded between 0 and 1. The model parameters α_{vG} and $n_{vG} > 1$ correspond to the inverse of the air entry suction and the pore-size distribution, respectively (Van Genuchten, 1980). The Macaulay bracket, $\langle \cdot \rangle$, is used to handle saturation states: when $h \geq 0$, $\langle -h \rangle = 0$, ensuring $\theta = \theta_s$ and $S_e = 1$ (saturated condition); conversely, when $h < 0$, $\langle -h \rangle = -h$, producing a value of θ between θ_r and θ_s (unsaturated condition). Thus, this compact notation enables Eq. (3) to unify saturated and unsaturated scenarios.

The Darcy seepage velocity, $\mathbf{q}_w$, is governed by the extended Darcy's law for unsaturated flow, and the hydraulic conductivity was derived by Van Genuchten (1980) as

$$\mathbf{q}_w = -k(h) \cdot (\nabla h - \mathbf{n}_g), \quad k(h) = k_{\text{sat}} \sqrt{S_e} \left[1 - \left(1 - S_e^{1/m_{vG}}\right)^{m_{vG}}\right]^2, \quad (4)$$

where $\mathbf{n}_g = \mathbf{g}/\|\mathbf{g}\|$ is the unit gravity vector ($\mathbf{n}_g = [0, -1]^T$ for two-dimensional problems and $\mathbf{n}_g = [0, 0, -1]^T$ for three-dimensional problems), k_{sat} represents the saturated hydraulic conductivity, and $m_{vG} = 1 - (1/n_{vG}) > 0$ is a model parameter (Van Genuchten, 1980). It should be noted that the isotropic hydraulic conductivity k is formulated as a function of the pressure head h because it depends on the effective saturation S_e , which is itself determined by h through Eq. (3).

Regarding the Richards equation, various equivalent formulations have been proposed in the literature. The mixed form presented in Eq. (2) is preferred over the standard head-based form because numerical schemes based on the mixed form are known to possess excellent mass conservation properties (Celia et al., 1990), a point that will be briefly

discussed later in Eq. (9). In the standard head-based formulation, applying the chain rule to the time derivative term $\partial\theta/\partial t$ yields

$$\frac{\partial \theta}{\partial t} = \underbrace{\frac{d\theta}{dh}}_{C(h)} \frac{\partial h}{\partial t}, \quad (5)$$

where $d\theta/dh$ is defined as the specific moisture capacity or the retention water capacity, denoted by $C(h)$. By differentiating the van Genuchten model in Eq. (3) with respect to h , an explicit expression for $C(h)$ is derived as

$$C(h) = \alpha_{vG}(n_{vG} - 1)(\theta_s - \theta_r) S_e^{1/m_{vG}} (1 - S_e^{1/m_{vG}})^{m_{vG}}. \quad (6)$$

In the van Genuchten model, $C(h)$ is always positive, and it would approach 0 when S_e is close to 0 (i.e., $-h$ is a very large positive number) or S_e is close to 1 (i.e., h is close to 0).

In order to solve Eq. (2) in both space and time, a standard finite element scheme is employed for spatial discretization, coupled with a backward finite difference scheme for time discretization. The derived residual vector can be expressed as

$$\mathcal{R}^{n+1,m} = \int_V \mathbb{N}^T \frac{\theta^{n+1,m} - \theta^n}{\Delta t} dV - \int_V \mathbb{E}^T \mathbf{q}_w^{n+1,m} dV - \underbrace{\int_{\Gamma_q} \mathbb{N}^T \bar{q}_h dA - \int_V \mathbb{N}^T \mathfrak{S} dV}_{\mathcal{G}^{n+1}}, \quad (7)$$

where the superscripts $n+1$ and m represent the time step level and iteration counter in the current time step, respectively, $\Delta t = t^{n+1} - t^n$ is the time step, $\mathbb{N}$ is the shape function matrix of the hydraulic pressure head, $\mathbb{E} = \nabla \mathbb{N}$ denotes the gradient of the shape function matrix, and $\mathcal{G}^{n+1}$ aggregates contributions from source/sink terms and Neumann (flux) boundary conditions prescribed in Γ_q . In poromechanics literature, **inward** flux across the boundary is taken as positive, whereas outward flux is assigned to be negative, which could be readily seen from the following boundary condition

$$-\mathbf{q}_w \cdot \mathbf{n} = \bar{q}_h \quad \text{on} \quad \Gamma_q. \quad (8)$$

As mentioned before, adoption of the standard head-based form would render the first term of the residual as

$$\int_V \mathbb{N}^T C(h^{n+1,m}) \frac{h^{n+1,m} - h^n}{\Delta t} dV = \int_V \mathbb{N}^T \frac{\theta^{n+1,m} - \theta^n}{\Delta t} dV + \text{Approximation errors (Lagrange Mean Value Theorem)}, \quad (9)$$

which does not strictly coincide with the current term in Eq. (7) and will lead to error accumulation and non-conservative results (Ju and Kung, 1997).

As $\mathcal{R}^{n+1,m}$ is a nonlinear function of the hydraulic pressure head h , Newton's method is adopted to compute the nodal increment vector $\Delta \mathbf{h}^{n+1,m+1} = \mathbf{h}^{n+1,m+1} - \mathbf{h}^{n+1,m}$ as

$$\mathcal{K}^{n+1,m} \Delta \mathbf{h}^{n+1,m+1} = -\mathcal{R}^{n+1,m}, \quad (10)$$

where the tangential operator matrix $\mathcal{K}^{n+1,m}$ is defined as (assuming $\mathcal{G}^{n+1}$ is independent of the hydraulic pressure head, otherwise, its contribution to $\mathcal{K}^{n+1,m}$ should also be taken into consideration)

$$\mathcal{K}^{n+1,m} = \frac{1}{\Delta t} \int_V \mathbb{N}^T C(h^{n+1,m}) \mathbb{N} dV + \int_V \mathbb{E}^T \mathbf{k}(h^{n+1,m}) \mathbb{E} dV + \int_V \mathbb{E}^T \left[\frac{\partial \mathbf{k}}{\partial h} \right]_{h^{n+1,m}} \cdot (\nabla h^{n+1,m} - \mathbf{n}_g) \mathbb{N} dV, \quad (11)$$

where $\mathbf{k}(h^{n+1,m}) = k(h^{n+1,m}) \mathbf{1}$ and $\mathbf{1}$ is the identity tensor. As we can see from Eq. (11), $\mathcal{K}^{n+1,m}$ is in general non-symmetric due to the last term, which considers the dependence of $k(h^{n+1,m})$ on the hydraulic pressure head. Many studies have ignored this term, especially for large-scale problems, as many iterative methods for solving large-scale linear system of equations such as the Preconditioned Conjugate Gradient (PCG) method require the coefficient matrix to be symmetric positive definite. Nevertheless, ignoring this term would lead to an increase in

the number of Newton iterations per time step, and the convergence rate would not be asymptotically quadratic anymore. Lastly, to solve the new hydraulic pressure head at the end of the time increment, it is necessary to know the hydraulic head at the start of the increment. Therefore, the initial conditions $h = h(x, t = 0)$ must be met to perform a seepage analysis.

The above residual vector $\mathcal{R}$ and tangential operator matrix $\mathcal{K}$ are for transient conditions, while for steady-state conditions, the fluxes entering and leaving a control volume are always the same when there are no source or sink terms and the hydraulic head does not depend on time (Yan et al., 2025). Therefore, the residual vector is updated as (where only the iteration counter m would appear and the subscript ss is used to emphasize the steady-state condition)

$$\mathcal{R}_{ss}^m = - \int_V \mathbb{E}^T \mathbf{q}_w^m dV + \mathcal{G}, \quad (12)$$

and the tangential operator matrix could also be easily modified as

$$\mathcal{K}_{ss}^m = \int_V \mathbb{E}^T \mathbf{k}(h^m) \mathbb{E} dV + \int_V \mathbb{E}^T \left[\frac{\partial \mathbf{k}}{\partial h} \Big|_{h^m} \cdot (\nabla h^m - \mathbf{n}_g) \right] \mathbf{N} dV. \quad (13)$$

In this work, three Gauss integration points were used in the evaluation of residual vector and tangential operator matrix.

2.2. Finite element lower bound limit analysis

The formulation of finite element lower bound limit analysis is briefly described herein, comprising two principal components: (1) the effective stress equation, since the yield constraints are enforced on the effective stress σ' (Sloan, 2013); (2) the formulation of a convex optimization problem that adheres to the disciplined convex programming (DCP) ruleset (CVX Research, Inc., 2012; Boyd and Vandenberghe, 2004).

2.2.1. Effective stress equation and yield function of unsaturated soils

For unsaturated soils, modeling shear strength is much more complicated than that of saturated soils and remains an active area of contemporary research. The most sophisticated Bishop's effective stress for unsaturated soils (Bishop, 1959) is given as

$$\sigma' = \sigma + \chi p_w \mathbf{1}, \quad (14)$$

where σ is the total stress tensor, χ is known as the Bishop effective stress parameter, p_w is the pore water pressure, and the air pressure is assumed negligible $p_a = 0$ (Song and Borja, 2014), so it does not appear in the above effective stress formula. The matric suction stress is defined as $s = p_a - p_w = -p_w > 0$ within the unsaturated zone, where the pore water pressure $p_w < 0$. In addition, Eq. (14) adopts the sign convention of continuum mechanics (tensile stresses are taken as positive), rather than the sign convention of soil mechanics.

Bishop (1959) postulated that the shear strength of unsaturated soils is governed by the single effective stress variable σ' . Nevertheless, the inherent limitations of this approach have been critically reviewed by Sheng et al. (2008). The primary drawback lies in the fact that the effective stress parameter χ lacks uniqueness: it is not a material constant but depends heavily on the current material state and the stress path. Consequently, the use of two independent stress state variables, net stress σ^{net} and matric suction s (Fredlund and Morgenstern, 1977), has gained widespread adoption in engineering practice owing to its practical utility (Sheng et al., 2008). Similarly, Fredlund et al. (1978) extended the shear strength equation by introducing the parameter ϕ^b , interpreted as an angle governing the rate of strength increase with respect to suction. Furthermore, the shear strength equation may also affect the SWCC (Fredlund et al., 1996). Despite these advances, implementing the dual-stress variable framework or the ϕ^b approach within the context of lower bound limit analysis remains mathematically challenging. To maintain compatibility with standard limit theorems,

the present work retains Bishop's effective stress definition, leaving the implementation of alternative frameworks for future research.

Given the aforementioned constraints associated with χ , certain simplifying assumptions are introduced to make the quantification feasible. In this study, two different models for χ are evaluated and contrasted. The first model sets $\chi = S_e$, where S_e is defined in Eq. (3); this formulation was proposed by Vanapalli et al. (1996) and Vanapalli and Fredlund (2000), and aligned with the poromechanics mixture theory derivations, given by Borja (2006) and Coussy and Monteiro (2007). The second model, originally proposed by Khalili and Khabbaz (1998) and reviewed by Khalili et al. (2004) with new evidence, has been applied to many studies (Khalili et al., 2008; Loret and Khalili, 2002; Khalili and Zargarbashi, 2010; Khalili and Loret, 2001). The expression of χ is given as

$$\chi = \begin{cases} \left(\frac{s}{s_e} \right)^{-0.55} & \text{if } s > s_e \\ 1 & \text{if } s \leq s_e \end{cases}, \quad (15)$$

where s_e is a suction value marking the transition between saturated and unsaturated states; for drying process, s_e is equal to the air entry value, which is governed by the grain size distribution (Khalili et al., 2004). Therefore, in order to use this model, we must know s_e a priori, and according to Khalili et al. (2004) and Khalili et al. (2008), s_e exhibits considerable variability among different samples (ranging from the order of 10 kPa to the order of 1000 kPa). Notably, if the calculated matric suction s is consistently lower than s_e , Eq. (15) yields a constant value of $\chi = 1$. In this scenario, the effective stress definition coincides with Terzaghi's effective stress. Consequently, we will also compare our results with the case where $\chi = 1$ in subsequent calculations.

Finally, based on the Bishop's effective stress σ' , and by assuming plane strain conditions, the Mohr–Coulomb yield function F_{MC} is expressed as a function of σ' as

$$F_{MC}(\sigma') = (\sigma'_x - \sigma'_y)^2 + (2\tau'_{xy})^2 - [2c' \cos \phi' - (\sigma'_x + \sigma'_y) \sin \phi']^2 \leq 0. \quad (16)$$

Here, c' and ϕ' represent the effective cohesion and internal friction angle, respectively. The term σ'_z is omitted from the yield function because, under plane strain conditions, it typically acts as the intermediate principal stress, which does not influence the Mohr–Coulomb yield criterion. Additionally, the notation τ'_{xy} is used to emphasize the effective stress measure, although for shear components, $\tau'_{xy} = \tau_{xy}$. Eq. (16) could also be expressed as a circle $X_{mc}^2 + Y_{mc}^2 = R_{mc}^2$ by letting

$$X_{mc} = \sigma'_x - \sigma'_y, \quad Y_{mc} = 2\tau'_{xy}, \quad R_{mc} = 2c' \cos \phi' - (\sigma'_x + \sigma'_y) \sin \phi'. \quad (17)$$

In the following lower bound limit analysis, if we want to formulate as a linear program (LP) problem, an inscribed regular polygon approximation will be employed to represent the above circular yield surface $X_{mc}^2 + Y_{mc}^2 = R_{mc}^2$, as proposed by Sloan (1988). However, the formulation can be directly cast as a second-order cone program (SOCP) problem without approximation, but in that case, we need to rewrite Eq. (16) in an equivalent form. This is because for current Eq. (16), F_{MC} is not a convex function of σ' by checking the positive definiteness of the Hessian matrix $\mathbf{H}_{F_{MC}}$

$$\mathbf{H}_{F_{MC}} = \begin{bmatrix} 2 \cos^2 \phi' & -2(1 + \sin^2 \phi') & 0 \\ -2(1 + \sin^2 \phi') & 2 \cos^2 \phi' & 0 \\ 0 & 0 & 8 \end{bmatrix}. \quad (18)$$

It is readily observed that $\mathbf{H}_{F_{MC}}$ is positive semidefinite if and only if $\phi' = 0$ ($0 \leq \phi' \leq \pi/2$), a special case of the Mohr–Coulomb yield criterion, known as the Tresca yield criterion. The equivalent convex form is given as

$$\tilde{F}_{MC}(\sigma') = \sqrt{(\sigma'_x - \sigma'_y)^2 + (2\tau'_{xy})^2} - [2c' \cos \phi' - (\sigma'_x + \sigma'_y) \sin \phi'] \leq 0. \quad (19)$$

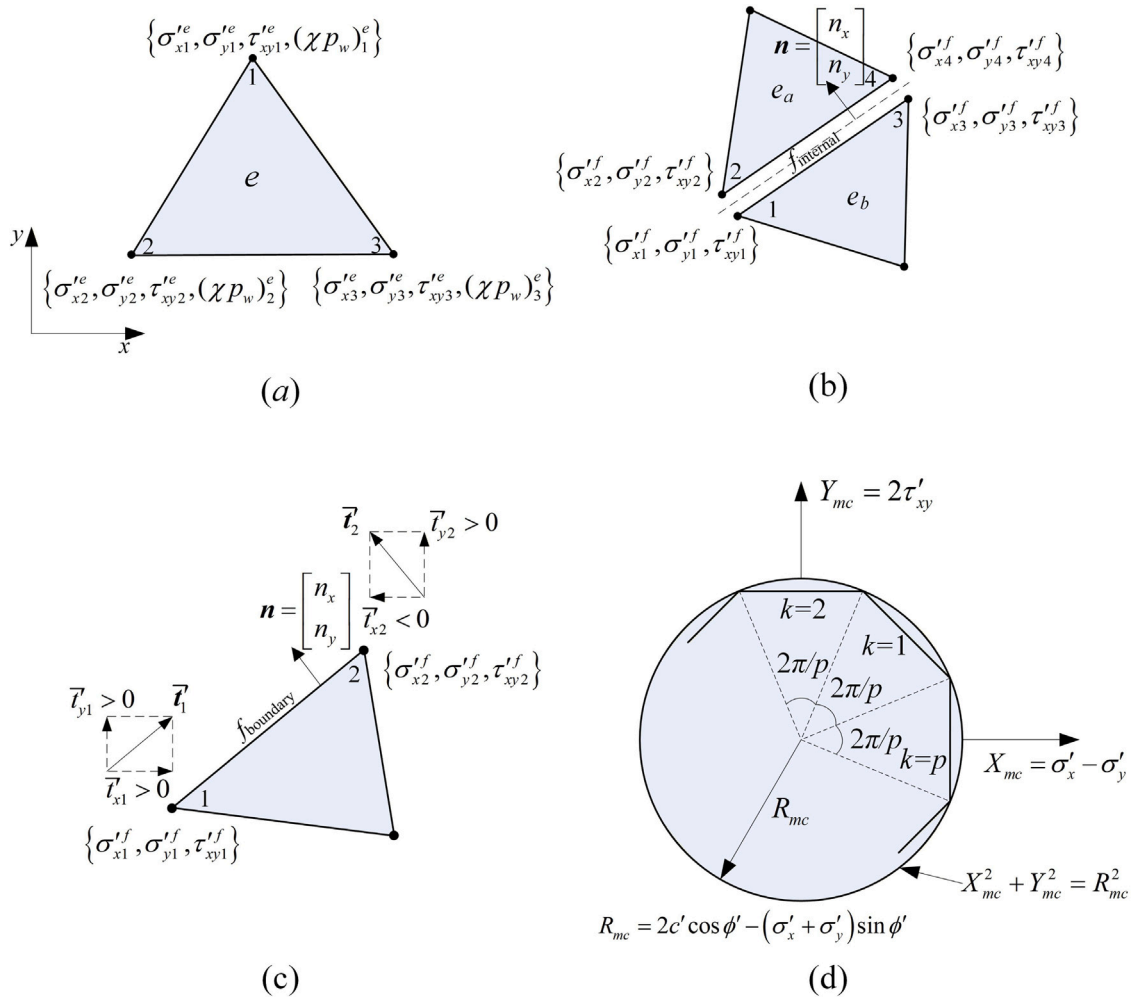


Fig. 1. Schematic diagrams for finite element lower bound limit analysis. The four subfigures correspond to different types of constraints described in Section 2.2.2: (a) is for element stress equilibrium constraint; (b) is for discontinuity equilibrium constraint; (c) is for traction boundary condition constraint; and (d) is for yielding condition constraint. Note that χp_w is continuous across the shared node of different elements, so it would not appear in some constraints explicitly.

The convexity of $\tilde{F}_{MC}$ will be shown in Section 2.2.2 using the DCP ruleset.

2.2.2. Disciplined convex programming in limit analysis

In this part, we will show how to formulate the lower bound limit analysis using disciplined convex programming (DCP) ruleset (Boyd and Vandenberghe, 2004), so it could be readily solved by CVX, a package for specifying and solving convex problems (CVX Research, Inc., 2012; Grant and Boyd, 2008). We refer the reader to Sloan (1988) for basic notation and explanation of the lower bound limit analysis, which we will not repeat here.

The first type of constraint in the lower bound limit analysis is the so-called element stress equilibrium constraint, as shown in Fig. 1(a), whereas for each element (represented by the superscript “e”), we have

$$A_{\text{equil}}^e X_{\sigma}^e = b_{\text{equil,h}}^e + b_{\text{equil,g}}^e. \quad (20)$$

This is an equality constraint, where both sides are affine, so it is valid according to the DCP ruleset. In the above equation, A_{equil}^e is a 2×9 matrix given as

$$A_{\text{equil}}^e = \frac{1}{2A^e} \begin{bmatrix} b_1^e & 0 & c_1^e & b_2^e & 0 & c_2^e & b_3^e & 0 & c_3^e \\ 0 & c_1^e & b_1^e & 0 & c_2^e & b_2^e & 0 & c_3^e & b_3^e \end{bmatrix}, \quad (21)$$

and the coefficients in the above matrix depend on the element nodal coordinates as (Sloan, 1988)

$$b_i^e = y_j^e - y_k^e, \quad c_i^e = -x_j^e + x_k^e, \quad 2A^e = x_1^e(y_2^e - y_3^e) + x_2^e(y_3^e - y_1^e) + x_3^e(y_1^e - y_2^e), \quad (22)$$

in which the indices i, j, k form an even permutation of 1, 2, 3. The vector of stress unknown of the current element X_{σ}^e could be written as

$$X_{\sigma}^e = [\sigma_{x1}^e, \sigma_{y1}^e, \tau_{xy1}^e, \sigma_{x2}^e, \sigma_{y2}^e, \tau_{xy2}^e, \sigma_{x3}^e, \sigma_{y3}^e, \tau_{xy3}^e]^T. \quad (23)$$

The right hand side of Eq. (20) accounts for the contributions from pore pressure field and gravity as

$$b_{\text{equil,h}}^e = \frac{1}{2A^e} \begin{bmatrix} b_1^e(\chi p_w)_1^e + b_2^e(\chi p_w)_2^e + b_3^e(\chi p_w)_3^e \\ c_1^e(\chi p_w)_1^e + c_2^e(\chi p_w)_2^e + c_3^e(\chi p_w)_3^e \end{bmatrix}, \quad b_{\text{equil,g}}^e = \begin{bmatrix} 0 \\ \gamma \end{bmatrix}. \quad (24)$$

As mentioned by Sloan (2013), during the limit analysis calculations, the pore pressure field is fixed, that is why χp_w will be moved to the right hand side in the constraints of the convex optimization problem, which is different from the hydro-mechanical coupled analysis for unsaturated porous media (Wang and Song, 2020). In this study, it is assumed that the unit weight of the soil γ is constant in both saturated

and unsaturated zones. However, it should be noted that the proposed optimization framework is capable of incorporating a spatially variable unit weight.

For the second type of constraints, we will combine the discontinuity equilibrium and boundary condition constraints proposed in Sloan (1988), and name it as face traction continuity constraints (represented by the superscript “ f ”). The mathematical form is given as

$$A_{\text{equil}}^f X_{\sigma}^f = b_{\text{equil}}^f. \quad (25)$$

Similar to Eq. (20), Eq. (25) is also a valid constraint in CVX according to the DCP ruleset. To facilitate matrix definitions in the above constraint, we define a traction projection matrix P associated with a face normal vector $\mathbf{n} = [n_x, n_y]^T$ as

$$P(\mathbf{n}) = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix}. \quad (26)$$

Additionally, a shear projection row vector $r(\mathbf{n})$, derived from the geometric transformation of the stress tensor, is introduced

$$r(\mathbf{n}) = [-n_x n_y \quad n_x n_y \quad n_x^2 - n_y^2]. \quad (27)$$

Now we provide the matrix definitions in Eq. (25) under four different scenarios:

1. When the face is an internal face, as shown in Fig. 1(b), we have

$$A_{\text{equil}}^f = \begin{bmatrix} P & -P & 0 & 0 \\ 0 & 0 & P & -P \end{bmatrix}, \quad (28)$$

$$X_{\sigma}^f = \begin{bmatrix} \sigma'_{x1} & \sigma'_{y1} & \tau'_{xy1} & \sigma'_{x2} & \sigma'_{y2} & \tau'_{xy2} & \sigma'_{x3} & \sigma'_{y3} & \tau'_{xy3} & \sigma'_{x4} & \sigma'_{y4} & \tau'_{xy4} \end{bmatrix}^T, \quad (29)$$

$$b_{\text{equil}}^f = [0 \quad 0 \quad 0 \quad 0]^T. \quad (30)$$

2. When the face is a boundary face and the surface traction is known (including the free-traction boundary), as shown in Fig. 1(c), we have

$$A_{\text{equil}}^f = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}, \quad (31)$$

$$X_{\sigma}^f = \begin{bmatrix} \sigma'_{x1} & \sigma'_{y1} & \tau'_{xy1} & \sigma'_{x2} & \sigma'_{y2} & \tau'_{xy2} \end{bmatrix}^T, \quad (32)$$

$$b_{\text{equil}}^f = [\bar{t}'_{x1} \quad \bar{t}'_{y1} \quad \bar{t}'_{x2} \quad \bar{t}'_{y2}]^T. \quad (33)$$

3. When the boundary face is roller-supported, only the shear component of the traction vector is known to be zero. Although this condition is not explicitly depicted in Fig. 1, Fig. 1(c) can still be used for illustration. Under this condition, the traction vectors $\bar{t}'_1$ and $\bar{t}'_2$ are parallel to $\mathbf{n}$ with arbitrary magnitudes. Consequently, this yields only two constraints per boundary face in the tangential direction, given by

$$A_{\text{equil}}^f = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}, \quad (34)$$

$$X_{\sigma}^f = \begin{bmatrix} \sigma'_{x1} & \sigma'_{y1} & \tau'_{xy1} & \sigma'_{x2} & \sigma'_{y2} & \tau'_{xy2} \end{bmatrix}^T, \quad (35)$$

$$b_{\text{equil}}^f = [0 \quad 0]^T. \quad (36)$$

4. When the boundary face is clamped in both directions, no constraint is needed for X_{σ}^f , since we assume that the traction at the fixed end can be arbitrary.

In the context of slope stability analysis, a non-zero equilibrium force vector, b_{equil}^f of Eq. (33), is required to define a valid optimization

objective. To satisfy this, a hypothetical vertical surcharge of one unit (1 kPa here) is applied at the top of the slope. The analysis is then treated as a collapse load maximization problem, similar to bearing capacity analysis (Sloan, 2013; Lyamin and Sloan, 2002). The goal is to identify the strength reduction factor (SRF) for which the computed collapse surcharge load vanishes (approaches zero). Failing to include this load reduces the analysis to a pure feasibility problem with no objective function, thereby precluding the direct calculation of the safety factor.

The last type of constraint is from Eq. (19), which could be reformulated as a second-order cone

$$\|A_i^{MC} X_{\sigma}^i\|_2 \leq (c_i^{MC})^T X_{\sigma}^i + d_i^{MC}, \quad i = 1, \dots, n_{\text{node}}, \quad (37)$$

where $n_{\text{node}} = 3 \times n_{\text{elm}}$ and n_{elm} is the total number of elements, the other matrices' definitions are given as

$$A_i^{MC} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad X_{\sigma}^i = \begin{bmatrix} \sigma'_{xi} & \sigma'_{yi} & \tau'_{xyi} \end{bmatrix}^T, \quad (38)$$

$$c_i^{MC} = [-\sin \phi' \quad -\sin \phi' \quad 0]^T, \quad d_i^{MC} = 2c' \cos \phi',$$

According to the DCP ruleset, this is a less-than inequality constraint, which requires the left hand side to be convex and the right hand side to be concave. For Eq. (37), the left hand side is the composition of an affine expression with the convex norm function, which is convex (Boyd and Vandenberghe, 2004); the right hand side is an affine expression, which is both convex and concave. Therefore, Eq. (37) is a valid constraint in CVX, and it constitutes a second-order cone program (SOCP) problem as the objective is affine. Please note that using the method proposed by Sloan (1988), the whole analysis will constitute a linear program (LP) problem, in which Eq. (37) will be modified into an abstract form of $G X_{\sigma} \leq h$ (Boyd and Vandenberghe, 2004). More details could be found from Sloan (1988) and Fig. 1(d). In subsequent analysis, we demonstrate that formulating the slope stability problem as a SOCP problem greatly improves computational efficiency compared to the original LP formulation, running on a standard desktop computer.

Finally, through the assembly operation on the overall mesh, the optimization problem could be formally written as a SOCP as

$$\begin{aligned} & \text{maximize} \quad \lambda_F \\ & \text{subject to} \quad \left(\sum_{f=1}^{n_{\text{face}}} A_{\text{equil}}^f \right) \{X_{\sigma}\} = \underbrace{\lambda_F \left(\sum_{f=1}^{n_{\text{face}}} b_{\text{equil}}^{f(1)} \right)}_{\text{hypothetical unit loading}} + \underbrace{\left(\sum_{f=1}^{n_{\text{face}}} b_{\text{equil}}^{f(2)} \right)}_{\text{matrix suction}}, \\ & \quad \left(\sum_{e=1}^{n_{\text{elm}}} A_{\text{equil}}^e \right) \{X_{\sigma}\} = \sum_{e=1}^{n_{\text{elm}}} (b_{\text{equil,h}}^e + b_{\text{equil,g}}^e) \\ & \quad \|A_i^{MC} X_{\sigma}^i\|_2 \leq (c_i^{MC})^T X_{\sigma}^i + d_i^{MC}, \quad i = 1, \dots, n_{\text{node}} \end{aligned} \quad (39)$$

where n_{face} is the total number of faces in the mesh and $\{X_{\sigma}\}$ is a long vector of nodal stresses by concatenating all X_{σ}^i ($i = 1, \dots, n_{\text{node}}$). The length of $\{X_{\sigma}\}$ vector is $9 \times n_{\text{elm}}$. The summation sign $\sum_{f=1}^{n_{\text{face}}}$ and $\sum_{e=1}^{n_{\text{elm}}}$ in the above equation should be understood as an assembly operator, rather than a small matrix addition. We also provide the equivalent CVX MATLAB code snippet in Appendix A directly to deepen the reader's understanding. From the code, we can see that by using the CVX built-in function norms(X , 2, DIM), which computes Euclidean norms along the dimension DIM, we could express the inequality constraint of Eq. (39) in a very efficient way, without any loops. Regarding CVX solvers to solve the convex optimization problem, we use SDPT3 (Version 4.0) for SOCP and SeDuMi (Version 1.02) for LP. The default solver settings are adopted and never changed during all the calculations below. Numerical computations were performed using MATLAB R2025b with CVX (Version 2.2).

2.2.3. Stress recovery technique

During post-processing, plotting the stress distribution requires assigning a unique value to each “global node”, and this process is known as the stress recovery process. Here, we use the term “global node” to distinguish it from the “local node” used in the finite element lower bound limit analysis. Unlike a local node, which belongs exclusively to a single element, a global node is shared by all connected elements. The unique value is determined through an averaging operation of all the local nodes that have the same physical coordinates as the global node, which could be mathematically written as

$$\chi_{\sigma}^k = \frac{\sum_{i=1}^{n_{\text{node}}} \chi_{\sigma}^i \times 1_{\{\text{local node } i \text{ coincides with the global node } k\}}}{\sum_{i=1}^{n_{\text{node}}} 1_{\{\text{local node } i \text{ coincides with the global node } k\}}}, \quad (40)$$

where a different notation for stress χ_{σ}^k is adopted to emphasize that this is a global stress quantity, and $1_{\{\circ\}}$ is the indicator function, which takes 1 if the statement inside the curly bracket $\{\}$ is true, and 0 otherwise. As mentioned previously at the beginning of Section 2, the number of local nodes n_{node} is equal to $3 \times n_{\text{elm}}$, i.e., three times the number of triangular elements. Based on χ_{σ}^k , the yield function value at each global node can also be evaluated and its spatial distribution visualized, as shown in Sections 4 and 5.

3. Slope stability assessment framework

This section focuses on the overall assessment procedure. Section 3.1 outlines the slope geometry and boundary conditions used for both seepage and limit analyses. Next, Section 3.2 details the algorithmic structure of the nonlinear finite element program for unsaturated fluid flow, including the critical potential seepage face review functionality. Finally, Section 3.3 introduces the bisection method used to calculate the factor of safety (FoS).

3.1. Finite element model geometry and reservoir drawdown boundary conditions

The numerical simulations utilize the finite element mesh and geometry depicted in Fig. 2.

The computational domain is defined by the vertices A, B, C, E, F, and G. Across all simulations, the downstream boundaries EF and CE are maintained at a fixed total head of 10 m (given $L = 10$ m), while the foundation base boundary BC is consistently treated as impermeable (no-flow boundary).

Several distinct hydraulic scenarios are examined. The first scenario represents a transient rapid drawdown event shown in Fig. 2(a). In this case, the boundaries AB and AG are impermeable. The upstream slope boundary GF is prescribed as a time-dependent Dirichlet boundary of the total water head, which also functions as a potential seepage face Γ_{ps} . To simulate a rapid drawdown condition, the reservoir water level is lowered to the target elevation over a duration of 0.01 d. Therefore, extending the drawdown period from 0.01 d to several days or even longer would transition the system toward the normal drawdown case, which is not depicted here. The second scenario considers the (extremely) slow drawdown case, in which the reservoir water level and the internal slope water level change simultaneously and remain horizontal, as also shown in Fig. 2(a). This is a **trivial** steady-state condition, as the analytical solution for the pressure head is readily obtained as

$$h_{\text{slow drawdown}} = 20 - L - y, \quad (41)$$

where y represents the elevation head. The third scenario considers a **non-trivial** steady-state seepage condition shown in Fig. 2(b). Here, the upstream vertical boundary AB is assigned a fixed total head (Dirichlet boundary), while AG remains a no-flow boundary. The boundary GF serves as a drainage boundary and is included in the potential seepage face set Γ_{ps} . The fourth, which is also the last scenario, considers a

rainfall event, although it is not drawn explicitly in Fig. 2. The rainfall is modeled using a positive flux applied to boundaries AG and GF, which are likewise included in the potential seepage face set Γ_{ps} . The iterative procedure that addresses potential seepage faces is detailed in Section 3.2 and Algorithm 1. To complement the textual descriptions of the boundary conditions above, a clear summary for each major simulation case is also provided in Table 1.

Regarding the mechanical boundary conditions for limit analysis, surface traction constraints are applied to slope boundaries GF and AG based on the position of the phreatic line. For boundary segments located below the phreatic line, both effective normal stress and shear stress are assumed to be zero ($\sigma'_n = 0$ and $\tau = 0$). For segments situated above the phreatic line, boundary conditions are determined by the specific unsaturated soil properties and whether matric suction is accounted for. Here we consider two possibilities: one is that we ignore matric suction and simply assume $\sigma'_n = 0$ and $\tau = 0$; and another is that we consider matric suction contribution using Eq. (14). Since the total stress σ is 0 on the slope surface above the phreatic line, the prescribed traction vector $\vec{t}'$ is calculated as

$$\vec{t}' = \sigma' \cdot \mathbf{n} = (0 + \chi p_w \mathbf{1}) \cdot \mathbf{n} = \chi p_w \mathbf{n}. \quad (42)$$

Since p_w is negative within the unsaturated zone, this prescribed traction induces compressive stress, which should be beneficial for slope stability. For the lateral boundaries AB and CE, no specific constraint is imposed on the effective normal stress σ'_n , while the shear stress τ is enforced to be zero. For the bottom boundary, because both vertical and horizontal displacements are constrained to be zero, in the lower bound limit analysis, there will be no specific requirements on the value of σ'_n and τ .

3.2. Unsaturated fluid flow nonlinear finite element program structure

This subsection details the algorithmic structure of the nonlinear finite element program for unsaturated fluid flow. While Section 2.1 established the mathematical formulations for the residual vectors and tangential operator matrices, the implementation requires an additional iterative procedure to address potential seepage faces, as outlined in Algorithm 1. Therefore, Algorithm 1 explicitly incorporates the seepage boundary review process alongside the standard Newton–Raphson iterations.

It is crucial to define the scope of the potential seepage boundary, here denoted as Γ_{ps} . In this framework, Γ_{ps} encompasses not only standard drainage and rainfall boundaries but also specific head-controlled nodes where the prescribed total head drops below the elevation head (i.e., the y -coordinate in 2D analysis). It should be noted that while the current algorithm handles infiltration (rainfall) and drainage, evaporation boundary conditions are not considered in the present implementation.

The iterative adjustment described in Algorithm 1 continues until the state of all nodes on Γ_{ps} stabilizes. Convergence of the seepage face is achieved when two physical criteria are simultaneously met:

1. For nodes currently fixed at a specific pressure head (Dirichlet BC, typically $h = 0$), the computed inward fluid flux must not exceed the prescribed external supply (e.g., rainfall intensity). If the computed inward flux is greater than the available rainfall (or greater than zero for a pure drainage face), the node cannot sustain saturation and must revert to a flux boundary.
2. For nodes currently fixed at a specified flux (Neumann BC), the computed pore water pressure must be negative (suction). If the pressure becomes positive, a seepage face has developed, and the node must switch to a pressure boundary.

Together with Algorithm 1, Fig. 3 provides the whole finite element procedure flowchart, from which a clear solution path is explicitly formulated.

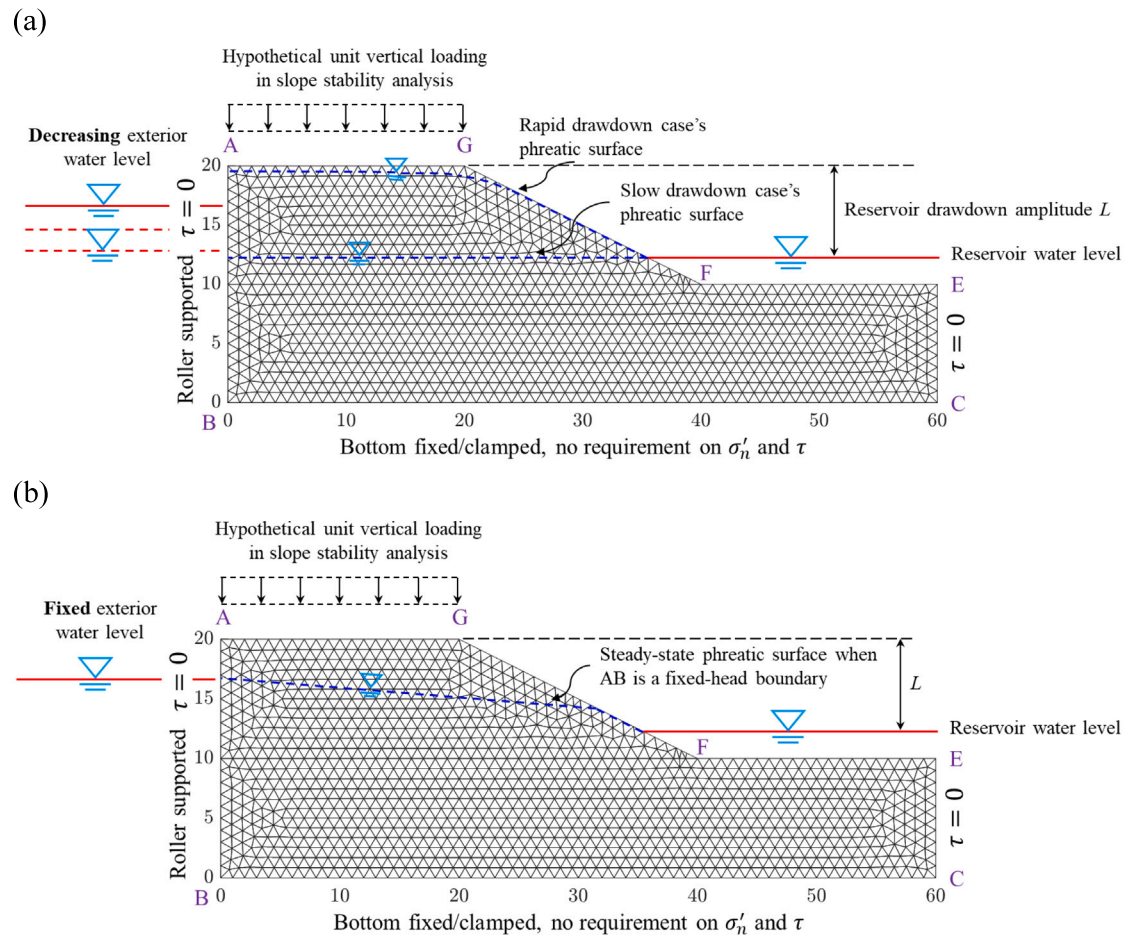


Fig. 2. Finite element model of unsaturated slope under variable water level: (a) Rapid transient and slow drawdown cases when AB is an impermeable boundary; (b) Non-trivial steady-state analysis cases when AB is a Dirichlet boundary with prescribed total head.

Table 1

Summary of boundary conditions across different hydraulic scenarios. Across all simulations, the downstream boundaries EF and CE are maintained at a fixed total head of 10 m (given $L = 10$ m), and the foundation base boundary BC is consistently treated as an impermeable boundary). Γ_{ps} refers to the potential seepage face set, which involves an iterative procedure to resolve, see Section 3.2 and Algorithm 1.

Boundary	(1) Rapid Drawdown	(2) Slow Drawdown	(3) Non-trivial Steady-State	(4) Post-drawdown Rainfall
AB	No-flow (Impermeable)	No-flow (Impermeable)	Fixed total head (Dirichlet)	No-flow (Impermeable)
AG	No-flow (Impermeable)	No-flow (Impermeable)	No-flow (Impermeable)	Positive flux $\bar{q}_h$ (Rainfall); included in Γ_{ps}
GF	Time-dependent Dirichlet (head lowered over 0.01 d); included in Γ_{ps}	Changing total head (trivial steady-state, $h = 20 - L - y$)	Drainage boundary; included in Γ_{ps}	Positive flux $\bar{q}_h$ (Rainfall); included in Γ_{ps}

3.3. Bisection algorithm of the strength reduction method

The safety factor of the slopes is evaluated using the strength reduction method in this study, along with the bisection algorithm. The inputs of the algorithm are the material parameters and the distribution of χp_w at each node. During the strength reduction process, χp_w is fixed according to [Sloan \(2013\)](#). The details are given in Algorithm 2, and the output is taken as the final factor of safety (FoS).

3.4. Other remarks

The assumption of passive air pressure $p_a = 0$ is justified here. In our current code development, we could not make p_a as an additional

primary variable, as in that case, we need to solve a two-phase flow problem, which is much more complicated than the current theoretical development. In addition, it should be noted that even a “rapid” drawdown still occurs over a finite duration rather than instantaneously, as an instantaneous drop would be physically unrealistic. In this work, the rapid drawdown process is modeled to be completed within 0.01 days. Regarding the rainfall infiltration, the maximum intensity considered in our study is 10 mm/h, which is a finite value rather than infinity. Consequently, we consider a reasonable assumption that the pore air pressure remains at zero throughout these processes. In fact, this assumption of passive air pressure $p_a = 0$ is widely adopted in the unsaturated soil mechanics community, see one of the milestone papers

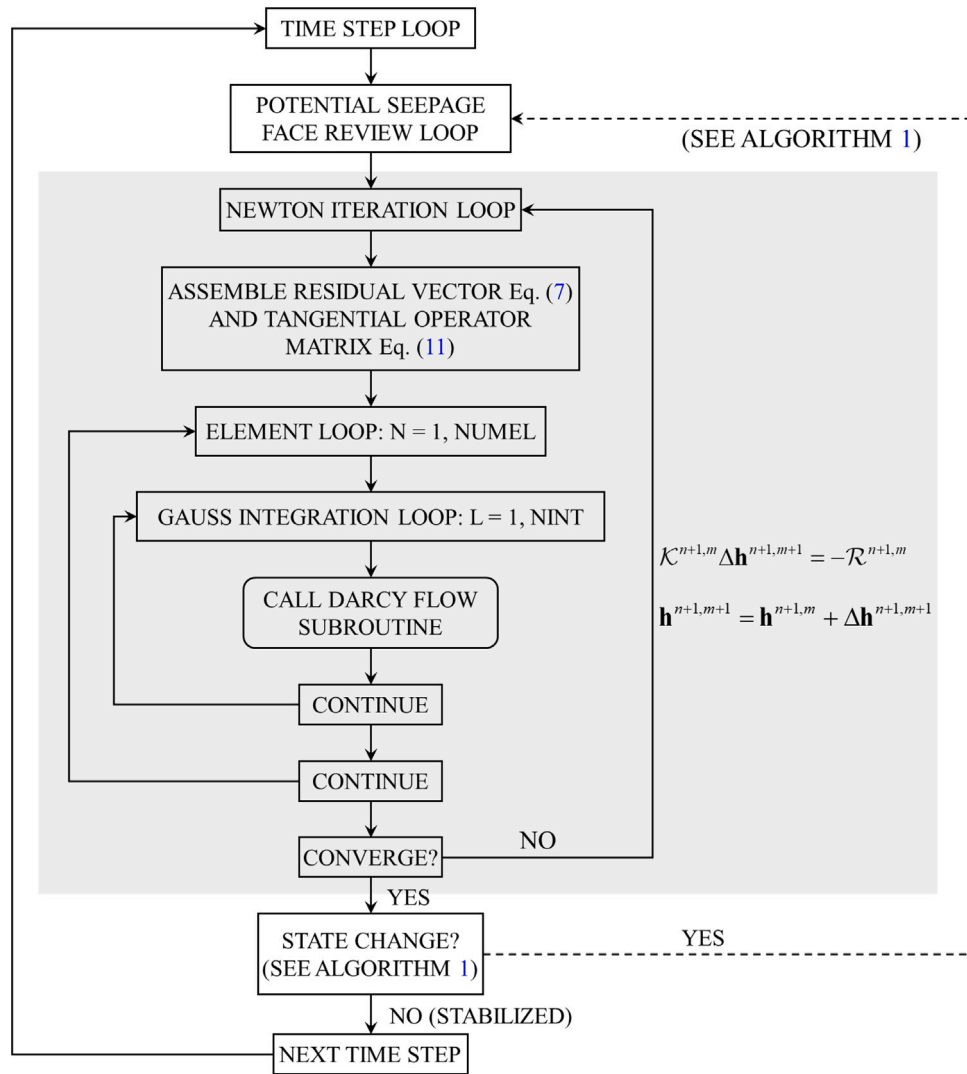


Fig. 3. Structure of a time-dependent unsaturated fluid flow nonlinear finite element program. As already revealed in the figure content, this flowchart should be interpreted in conjunction with Algorithm 1. For example, the shaded region corresponds to NEWTONSOLVER of Algorithm 1. The steady-state analysis structure is analogous to the transient case, distinguished only by the omission of the outer time step loop.

by Prof. Daichao Sheng (Sheng et al., 2008). The rapid drawdown scenario has also been investigated by Prof. D. V. Griffiths in Lane and Griffiths (2000), while the issue of air entrapment or positive pore-air pressure is also neglected.

Another remark concerns the time-integration scheme. To the best of the authors' knowledge, the backward Euler method should generally be the first choice in computational poromechanics due to its unconditional stability, unless highly accurate results are required in conjunction with quite large time steps. In such cases, higher-order methods would be more appropriate. However, for typical two-dimensional problems (e.g., those with a few thousand elements, which are not computationally prohibitive), the accuracy of the results can usually be improved simply by reducing the time step size. Both the backward Euler and Crank–Nicolson schemes belong to the category of one-step method. There are also linear multistep methods which may yield even better accuracy, see Borja (1991).

4. Model validation

4.1. Collapse pressure for a smooth strip footing

Before working on the slope stability problems, the proposed CVX-based approach is first validated on a bearing capacity problem for

cohesive-frictional soil. The original problem setting was given in Sloan (1988) and the same finite element mesh is adopted, as shown in Fig. 4. For this problem, we only need to solve Eq. (39) once without any bisection process, and the solutions are obtained using LP and SOCP formulations.

The analytical solution of bearing capacity provided by Prandtl (1920) is given as

$$q_u = c' \left[e^{\pi \tan \phi'} \tan^2 \left(\frac{\pi}{4} + \frac{\phi'}{2} \right) - 1 \right] \cot \phi' = c' N_c, \quad (43)$$

where N_c is known as the coefficient of cohesion bearing capacity, and the soil unit weight γ is ignored in the above formula (weightless soil). For $c' = 1$ kPa and $\phi' = 40^\circ$, i.e., the same parameters as those in Sloan (1988), the exact value of N_c is 75.357, and we will compare this exact value with the numerical lower bounds given in Table 2. Since $c' = 1$ kPa, the optimized λ_F from Eq. (39) will be equal to the numerical N_c . In the same Table 2, we also incorporate the original results calculated by Sloan (1988). From the table, under the LP formulation, the numerical N_c is sensitive to the number of sides p used in the linearized yield polygon of the original Mohr–Coulomb yield surface (see Fig. 1). In general, $p = 48$ is usually enough for most scenarios. In contrast, the SOCP formulation offers a huge advantage in that the parameter p is not needed in the formulation, as the yield

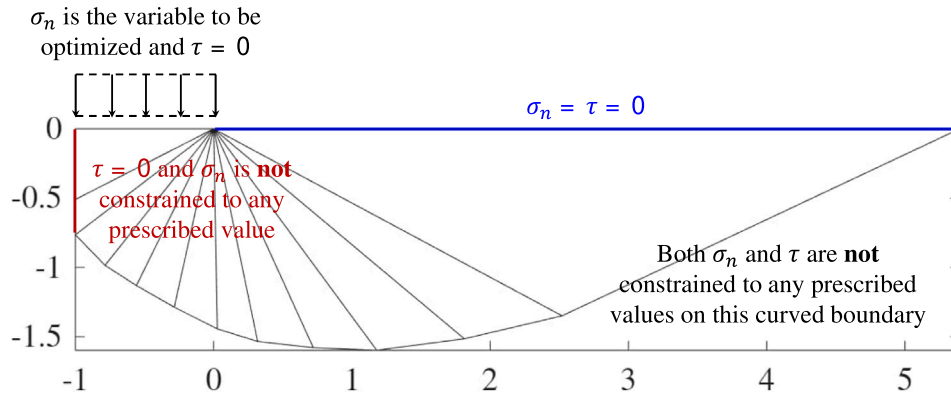


Fig. 4. Finite element mesh for a smooth strip footing on the weightless cohesive-frictional soil following Sloan (1988). Complete boundary conditions are given and there is no pore water pressure, which means the effective stress tensor σ' is the same as the total stress tensor σ .

Algorithm 1 Pseudo code of iterative solver for potential seepage boundary conditions

Require: Previous state $\mathbf{h}^n$, time increment Δt , potential seepage boundary node set Γ_{ps}

Ensure: Converged solution $\mathbf{h}^{n+1}$ satisfying seepage constraints

Initialization:

```

1: Assign a Flux-type boundary condition (Neumann) to all nodes in  $\Gamma_{ps}$ . The flux is computed from  $\mathbf{h}^n$ , which means it could be different from the prescribed value  $\bar{q}_{ext}$  (e.g., rainfall)
2:  $k \leftarrow 0$  ▷ Seepage iteration counter
3:  $StateChanged \leftarrow \text{true}$ 
Iterative Adjustment Loop:
4: while  $StateChanged$  is true do
5:    $StateChanged \leftarrow \text{false}$ 
6:   Solve nonlinear system using NEWTONSOLVER shown in Fig. 3 (shaded region) with initial guess  $\mathbf{h}^{n+1,0} = \mathbf{h}^n$  (or previous  $\mathbf{h}^{*k}$ ). Iterate until  $\|\mathcal{R}\| < \epsilon_{tol1}$  or  $\|\Delta \mathbf{h}\| < \epsilon_{tol2}$ .
7:   Let  $\mathbf{h}^{*k}$  be the converged solution from the Newton solver.
   // Check validity of boundary conditions using  $\mathbf{h}^{*k}$ 
8:   for each node  $i \in \Gamma_{ps}$  do
9:     if node  $i$  is Flux-Controlled (Neumann) then
10:      if  $h_i^{*k} > 0$  then ▷ Pressure is positive (saturated)
11:        Switch BC of node  $i$  to H-type (Dirichlet,  $h = 0$ )
12:         $StateChanged \leftarrow \text{true}$ 
13:      end if
14:    else if node  $i$  is Pressure-Controlled (Dirichlet) then
15:      Compute nodal flux  $\bar{q}_{h,i}$  based on  $\mathbf{h}^{*k}$  ▷ Positive indicates inward flux
16:      if  $\bar{q}_{h,i} > \bar{q}_{ext,i}$  then
17:        ▷ Computed inward flux exceeds available supply (rainfall)
18:        Switch BC of node  $i$  to Q-type (Neumann,  $\bar{q} = \bar{q}_{ext,i}$ )
19:         $StateChanged \leftarrow \text{true}$ 
20:      end if
21:    end if
22:  end for
23:   $k \leftarrow k + 1$ 
24: end while
25: Return  $\mathbf{h}^{n+1} \leftarrow \mathbf{h}^{*k}$ 

```

constraint is satisfied in an exact way. Using the stress recovery approach described in Section 2.2.3, Fig. 5 depicts the spatial distribution of the yield function $f(\sigma')$, and we can see the darkest red region (i.e., $f(\sigma') \approx 0$ implies the yield zone) perfectly follows the curved boundary, which is the assumed failure surface in the limit equilibrium method

Algorithm 2 Bisection method for determining the FoS in lower bound limit analysis

Require: Unit weight γ , effective cohesion c' , effective internal friction angle ϕ' , distribution of suction stress χp_w at nodes, initial bounds SRF_{min} and SRF_{max} , tolerance ϵ_{tol} (e.g., 0.001)

Ensure: Converged Strength Reduction Factor (SRF), which is assumed equivalent to FoS

Initialization:

```

1:  $iter \leftarrow 0$ 
Bisection Loop:
2: while  $SRF_{max} - SRF_{min} > \epsilon_{tol}$  do
3:    $iter \leftarrow iter + 1$ 
4:    $SRF_{trial} \leftarrow (SRF_{min} + SRF_{max})/2$ 
   // Apply strength reduction to Mohr-Coulomb parameters
5:    $c'_{red} \leftarrow c'/SRF_{trial}$ 
6:    $\phi'_{red} \leftarrow \arctan(\tan(\phi')/SRF_{trial})$ 
   // Call Optimization Module
7:   Solve the lower bound limit analysis optimization problem Eq. (39) using CVX with  $\chi p_w$  and reduced parameters  $c'_{red}$  and  $\phi'_{red}$  to maximize the load multiplier  $\lambda_F$ .
   // Check for early convergence
8:   if  $|\lambda_F| < \epsilon_{tol}$  then
9:     break ▷ Load multiplier is close enough to zero; exact FoS found
10:  end if
   // Update bisection bounds based on stability status
11:  if  $\lambda_F > 0$  then
12:    ▷ A valid statically admissible stress field exists; slope is stable
13:     $SRF_{min} \leftarrow SRF_{trial}$ 
14:  else
15:    ▷ Problem is infeasible or  $\lambda_F \leq 0$ ; slope is unstable
16:     $SRF_{max} \leftarrow SRF_{trial}$ 
17:  end if
18: end while
19: Return  $FoS \leftarrow (SRF_{min} + SRF_{max})/2$ 

```

(Chen and Scawthorn, 1970). Therefore, the optimized stress results of Eq. (39) have been verified.

4.2. Slope stability analysis of a dry slope

In the second verification example, the same dry slope as that in Tschuchnigg et al. (2015a) is considered, which is shown in Fig. 6. In fact, this combination of slope inclination angle $\alpha_s = 45^\circ$ and

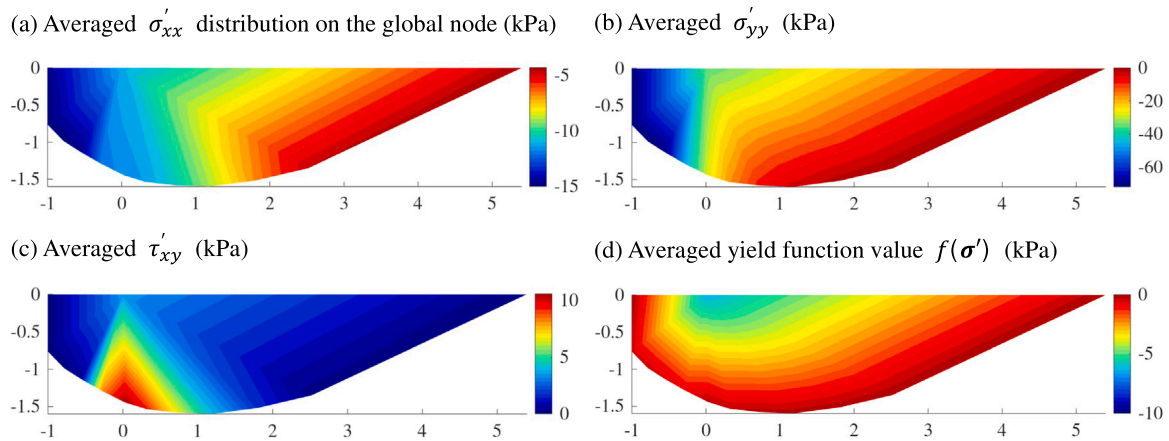


Fig. 5. Optimized stress tensor and yield function distributions for the smooth strip footing problem with SOCP formulation.

Table 2

Bearing capacity factor N_c results of smooth strip footing problem using different convex formulations.

Convex formulation	Num. of sides p in approx. MC	Bearing capacity factor N_c
LP	12	61.0236
	24	68.4005
	48	71.0862
	48 ^a	71.9223
SOCP	N.A.	71.8625

^a Reference results of Sloan (1988).

Table 3

Comparison of FoS and computational time for different formulations and meshes.

Convex formulation and mesh/geometry information	Factor of Safety (FoS)	Computational time (s)
LP ($p = 48$)/coarser mesh of Fig. 6(b)	1.3955	123.65
SOCP/coarser mesh of Fig. 6(b)	1.3975	51.787
SOCP/finer mesh of Fig. 6(a)	1.4180	261.8
SOCP/smaller geometry of Fig. 6(c)	1.4170	84.57
Tschuchnigg et al. (2015a)	1.45	N.A.

slope height $H = 10$ m has been widely used in other studies and simulations (Dawson et al., 1999). The main difference lies in the overall geometry of the model, specifically the outward extension distances beyond the slope crest and toe, as well as the downward extension depth beneath the slope base. As for the material parameters, we still assume a cohesive-frictional soil with $c' = 20$ kPa, $\phi' = 25^\circ$, and $\gamma = 20$ kN/m³. This example serves three distinct purposes: (1) to verify the accuracy of the proposed CVX-based approach and bisection algorithm; (2) to show the advantage of SOCP over LP in terms of computational efficiency; (3) to investigate the effects of model geometry and mesh size on the calculated FoS, as shown in Fig. 6.

Table 3 summarizes the calculated FoS and computational time for various scenarios. It can be seen that using FoS = 1.45 as a reference value (Tschuchnigg et al., 2015a), the largest error is below 4%, occurring for the case of a coarser mesh and when the LP formulation is adopted. The smallest error of 2.2% corresponds to the case of a finer mesh with the SOCP formulation, and the optimized stress distribution is plotted in Fig. 7. Again, we can see that the darkest red region is consistent with the numerical shear band (the region with high values of equivalent plastic strain) produced by displacement finite-element/finite-difference strength reduction techniques (Tschuchnigg

et al., 2015a; Dawson et al., 1999). Therefore, the correctness of our method has been confirmed.

Table 3 also records the computational time in the coarser mesh. It can be seen that the computational time for the SOCP formulation is less than half of that for the LP formulation. While one might contend that for such short computational time of less than 3 min, this level of acceleration is insignificant. However, as will be shown in the next example of the vertical cut problem, this efficiency enhancement becomes much more pronounced as the mesh becomes finer.

Regarding the effects of model geometry and mesh size, Table 3 reveals that a finer mesh generally yields a tighter lower bound, albeit at the expense of increased computational time. The two model geometries in Figs. 6(a) and 6(c) yield quite close FoS (the difference is 0.001), suggesting that the model geometry can be optimized as long as the failure pattern remains unaffected. Meanwhile, the original geometry given in Tschuchnigg et al. (2015a) also provides a rule of thumb to determine the model dimensions, ensuring that the calculated FoS is not distorted by an overly small model geometry. As shown in Fig. 2, the model geometries used in Section 5 are set to be sufficiently large to ensure that the failure surface is fully contained.

4.3. Vertical cut problem

The last verification example considers a vertical cut problem, an important practical problem for construction in undrained clay excavation. The finite element mesh is shown in Fig. 8, which consists of 5562 triangular elements, and it is sufficiently fine compared with the finest mesh used in Lyamin and Sloan (2002) (6400 elements) and Pastor et al. (2000) (3232 elements). For this problem, we need to convert the calculated FoS to the stability number N_s using the following formula

$$N_s = (H \times \text{FoS} \times \gamma) / c_u, \quad (44)$$

where H is the current vertical cut height, γ is the soil unit weight, and $c' = c_u$ is the undrained cohesion of clay. In the undrained analysis, the friction angle ϕ' will be set to zero and other parameters are given as $H = 10$ m, $\gamma = 15$ kN/m³ and $c_u = 75$ kPa.

As demonstrated in Table 4, this example highlights the superiority of our approach in balancing the trade-off between accuracy and computational time. Several conclusions can be drawn from the table. First of all, the “traditional” lower bound of FoS = $2c_u/(\gamma H) = 1$ (which leads to $N_s = 2$), commonly introduced in textbooks and graduate courses, is actually a very loose bound, because the stress field provided by Chen and Scawthorn (1970) is not fully optimized. By employing optimization techniques, significantly tighter lower bounds can be obtained, as shown by this table. Second, for this fine mesh, the high efficiency of the SOCP formulation over the LP formulation is highly evident (compared with Table 3); the computational time required for

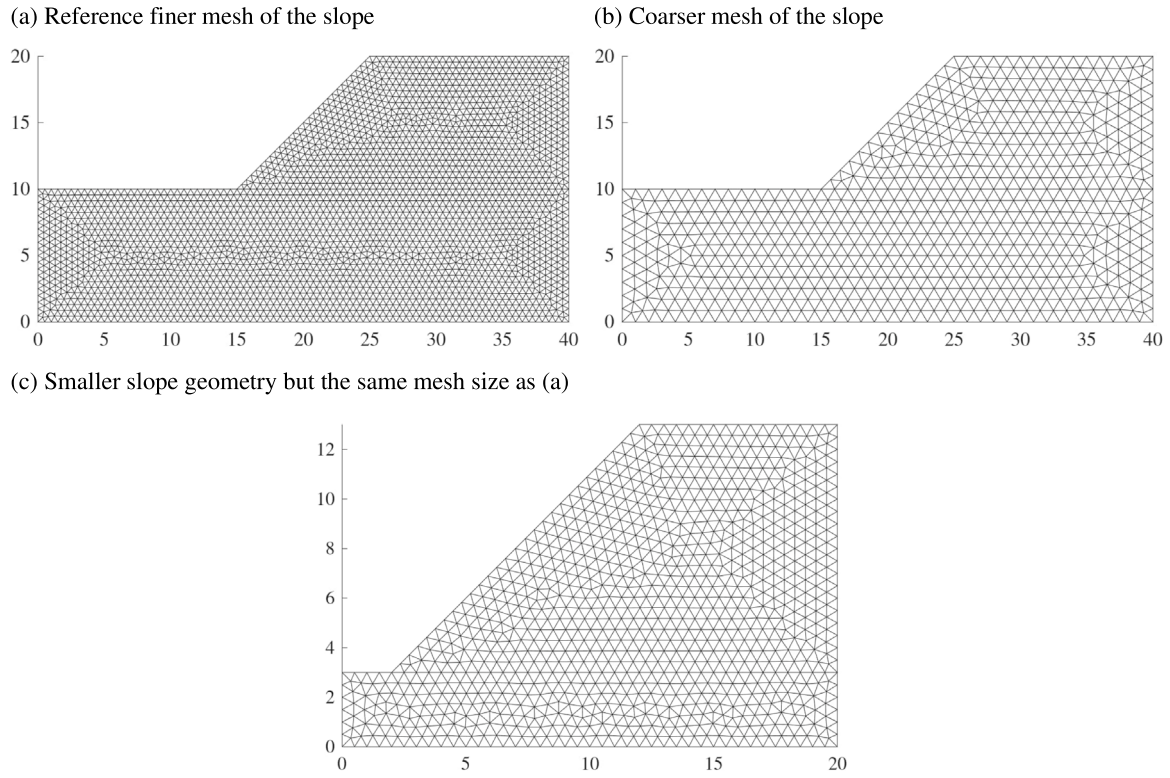


Fig. 6. Finite element mesh of a dry slope following Tschuchnigg et al. (2015a). Different finite element mesh sizes and slope geometries are also considered in this problem.

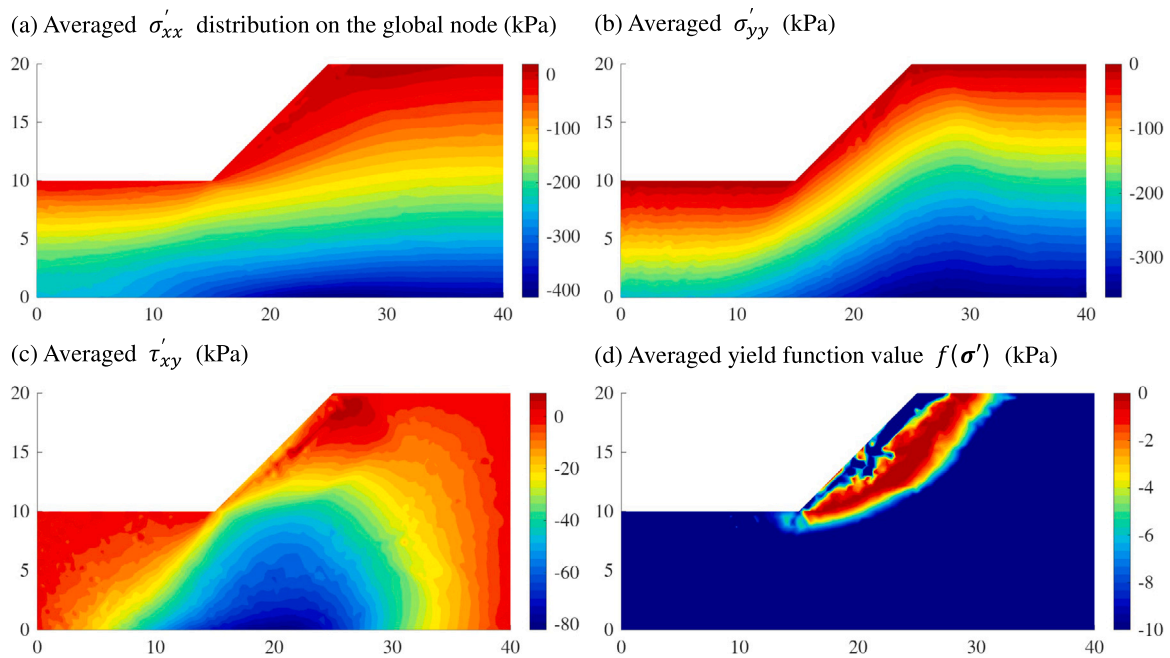


Fig. 7. Spatial distributions of the optimized stress field and yield function for the dry slope problem based on the SOCP formulation on a finer mesh, i.e., Fig. 6(a).

LP is nearly six times that of SOCP. Third, compared with the reference results, although our N_s is marginally smaller (by approximately 2%), our approach offers a massive advantage in computational efficiency. It requires only 2.5 min, which is in stark contrast to the 30,000 min reported by Pastor et al. (2000). Furthermore, unlike the algorithm in Lyamin and Sloan (2002), which is tailored to specific optimization problems, our solver (CVX) is highly versatile and capable of handling

a wide range of optimization tasks (Boyd and Vandenberghe, 2004). In addition, the optimized stress distribution is given in Fig. 9, and we can see that the dark red zone is consistent with the linear failure surface in the vertical cut problem (Chen and Scawthorn, 1970). Consequently, taking both accuracy and efficiency into account, our method demonstrates distinct advantages.

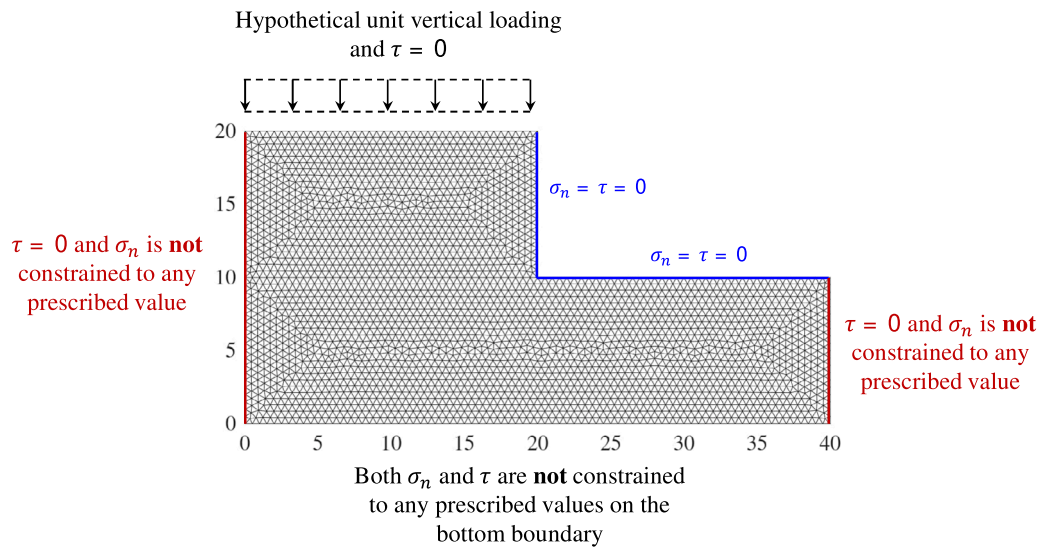


Fig. 8. Finite element mesh of the vertical cut problem, which consists of 5562 triangular elements. Please note that the phrase “not constrained to any prescribed values” does not refer to an unconstrained boundary condition; rather, it indicates that the stress at this point is not included in the constraint equations of the optimization problem. This interpretation also applies to other examples.

Table 4

Comparison of different formulations regarding stability number and computational features in the vertical cut problem.

Formulation	Num. of elements	Stability number N_s	Computational time	Feature
SOCp of CVX and bisection method	5562	3.7168	2.5167 min	Balance the trade-off between accuracy and computational time
LP of CVX and bisection method	5562	3.709	13.77 min	LP's weakness relative to SOCp is more apparent than in Table 3
Non-linear programming in Lyamin and Sloan (2002)	6400	3.772	17 min	Best known lower bound until now but algorithm is tailored to specific problems
Simplex linear programming in Pastor et al. (2000)	3232	3.76	3 weeks $\approx$ 30,000 min	Used to be the best lower bound before 2002 but computationally infeasible for real problems

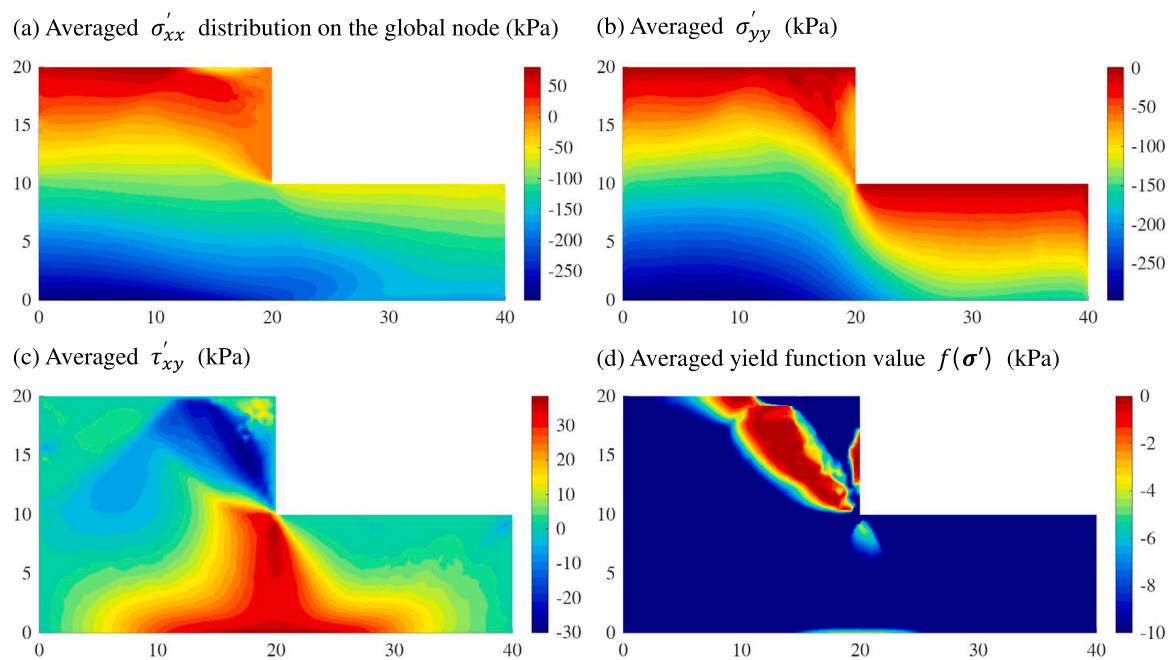


Fig. 9. Spatial distributions of the optimized stress field and yield function for the vertical cut problem based on the SOCp formulation.

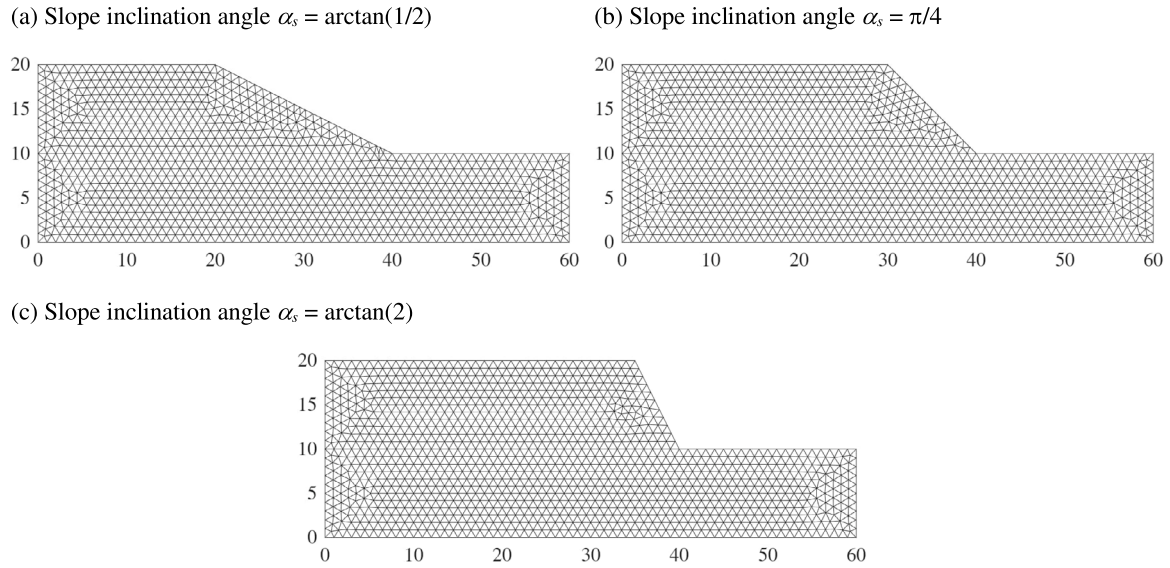


Fig. 10. Three different slope geometries and finite element meshes in model application. Boundary conditions are already given in Fig. 2.

5. Model application considering unsaturated flow

In the model application, the lower bound limit analysis of slope stability based on the SOCP formulation is integrated with the unsaturated flow. The overall framework and flowchart have already been mentioned in Section 3. To make our conclusion more broadly applicable across different slope geometries, three slope inclination angles are examined: $\alpha_s = \arctan(1/2)$, $\alpha_s = \pi/4$, and $\alpha_s = \arctan(2)$, as shown in Fig. 10. Because the seepage field is considered in the slope stability analysis, the effective stress equation Eq. (14) and the definition of χ will be of vital importance. In this work, four different choices will be tested, as listed below:

1. The suction contribution is ignored, which means that only positive pore water pressures are considered and all negative pore water pressures are set to zero. The effective stress equation would coincide with Terzaghi's effective stress equation.
2. The suction contribution is considered, so the traction boundary must be carefully addressed, which is already described in Section 3.1. For the parameter χ , we have three choices, according to Section 2.2.1, which are briefly listed here
 - (a) $\chi = S_e$ proposed in Vanapalli et al. (1996);
 - (b) χ is determined according to Eq. (15) with $s_e = 25$ kPa (Khalili and Khabbaz, 1998);
 - (c) $\chi \equiv 1$ when all matric suction values are lower than s_e . Please distinguish this choice from the choice of ignoring suction contribution. Although both share the same effective stress equation, they might have different boundary tractions and different $b_{\text{equil,h}}^e$ values according to Eq. (24). In other words, one choice would consider negative pore water pressure in $b_{\text{equil,h}}^e$ and the other one will not.

The mechanical and hydraulic parameters of the soil are listed in Table 5, which are representative values according to Zhou and Qin (2022) and Table 1 “Soil physical properties of the five example soils” of Van Genuchten (1980).

5.1. Slow drawdown scenario

The scenario of slow reservoir drawdown is initially examined, as in this case, the pore water pressure could be analytically obtained in Eq. (41) without any numerical techniques, so the seepage field is independent of discretization errors in space and time. To model the slow

Table 5

Soil mechanical and hydraulic parameters in model application.

Parameter	Value	Unit
Soil unit weight γ	20	kN/m ³
Effective cohesion c'	10	kPa
Effective internal friction angle ϕ'	20	deg
Saturated hydraulic conductivity k_{sat}	1.516×10^{-6}	m/s
vG model parameter α_{vG}	1.06	1/m
vG model parameter n_{vG}	1.395	1
Saturated water content θ_s (porosity)	0.469	1
Residual water content θ_r	0.106	1

drawdown process, 11 different water table heights are considered, ranging from 20 m to 10 m in decrements of 1 m. For each water table elevation, the FoS is calculated for the three slope inclination angles and the four aforementioned matric suction models. This procedure requires a total of $N_{\text{total run}} = 11 \times 3 \times 4 = 132$ executions of the bisection algorithm, and all results are displayed in Fig. 11. In the same figure, we also add the limit equilibrium method (LEM) solution using the Morgenstern-Price method (Morgenstern and Price, 1965) for a few parameter combinations (see the solid red star marker), which could serve as an indirect verification of our lower bound results.

The most obvious trend observed from Fig. 11 is that as the water table decreases (each plot should be interpreted from right to left), the FoS would generally exhibit a “U-shaped” curve, i.e., the FoS would first decrease and then increase, except for the case of very steep slope and the suction contribution is neglected, i.e., the black curve in Fig. 11(c). For Fig. 11(a), the trend of the black curve is consistent with the findings in Lane and Griffiths (2000) and the authors have provided a quite convincing explanation, which is rephrased here for completeness. The explanation says that during the early phases of water level reduction, the destabilizing impact of the increase in self-weight of the slope outweighs the gains in frictional resistance, leading to a decline in FoS. At higher drawdown levels, however, the enhancement in frictional strength becomes the dominant factor, resulting in a recovery of FoS (Lane and Griffiths, 2000). Note that for other curves with different colors, the increase in FoS at higher drawdown levels corresponding to low water table heights is mainly due to the matric suction.

When incorporating the contribution of matric suction, the three aforementioned options for Bishop effective stress parameter χ yield progressively higher Factor of Safety (FoS) values from option 2(a)

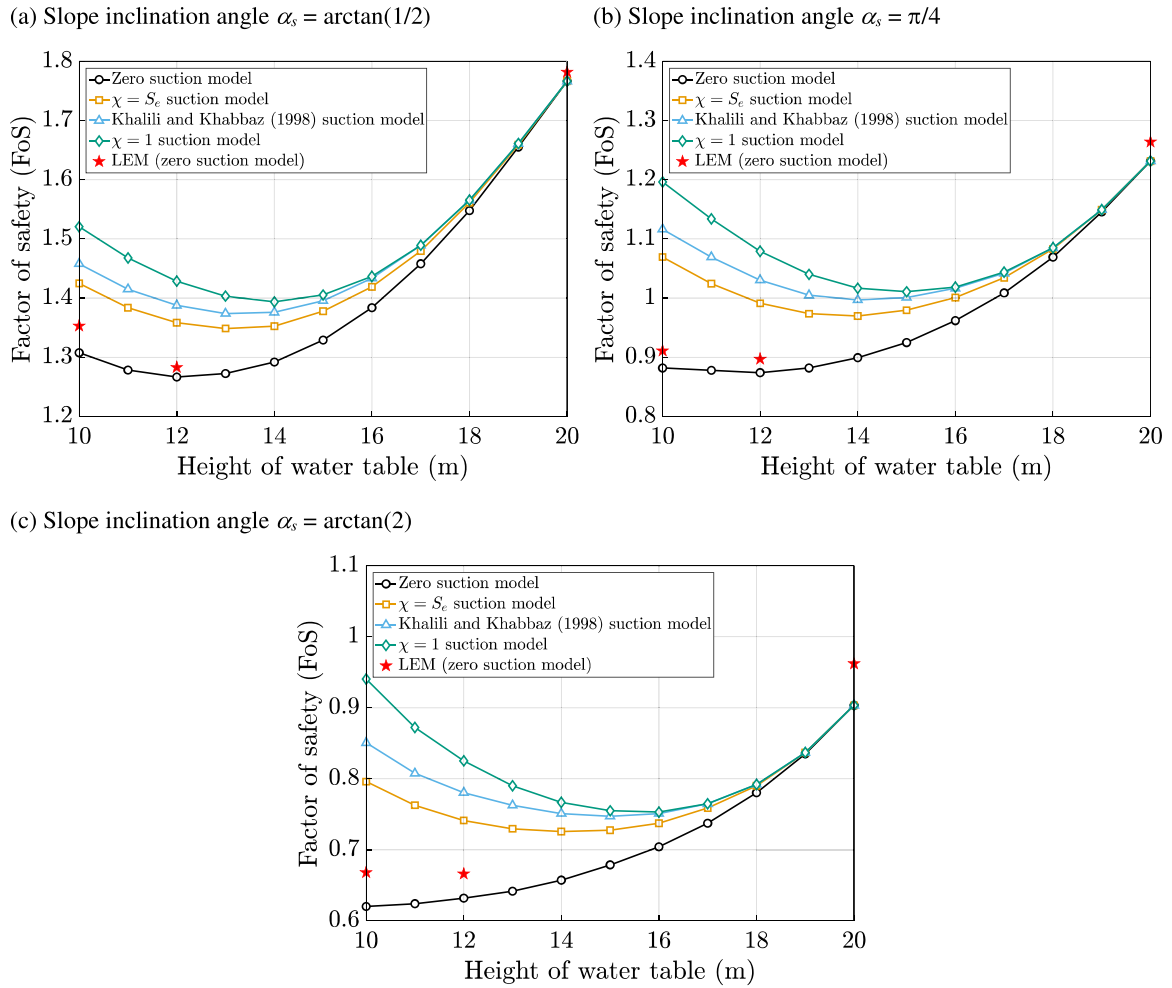


Fig. 11. Change of FoS with respect to the water table height for different slope inclination angles and different suction models under the slow drawdown scenario.

to option 2(c), rendering the stability estimations increasingly non-conservative. Moreover, the discrepancy among these different assumptions on χ is markedly more pronounced for steep slopes compared to gentle ones. Therefore, from a practical view, the formulation $\chi = S_e$ is recommended, and this form of χ is still the most widely used in the unsaturated soil mechanics community, despite some limitations as mentioned above in Section 2.2.1.

The matric suction would also affect the failure surface pattern for the slope with the smallest inclination angle $\alpha_s = \arctan(1/2)$. In Fig. 12, the spatial distribution of the yield function $f(\sigma')$ is displayed using our approach, and as a comparison, the LEM results obtained from SLOPE/W are also shown. It is initially observed that two approaches share similar characteristics: when the suction contribution is ignored, the failure surface would pass the slope toe; whereas once the suction contribution is accounted for, the failure pattern shifts to a basal failure. However, for steeper slopes, it is possible that only toe failure would occur, no matter whether we consider the suction contribution or not, as shown in Fig. 13 for the steepest slope. In this case, the shape of the dark red zone more closely resembles a linear failure surface rather than a circular failure surface.

5.2. Non-trivial steady-state seepage scenario

In terms of seepage analysis, the previous slow drawdown scenario belongs to the steady-state analysis, despite the solution being trivial. Consequently, this subsection investigates the effect of non-trivial

steady-state seepage on slope stability, and a schematic diagram has already been given in Fig. 2(b). We assume that the water level of the interior reservoir is fixed at 10m, while the exterior water level is maintained at 20m (assuming an external water supply), and similar to before, the FoS is calculated for the three slope inclination angles and the four aforementioned matric suction models. The LEM solution for the zero suction model is also added for reference.

The steady-state pore water pressure head h distributions for three slope inclination angles are displayed in Fig. 14. Since this is a steady-state seepage analysis, only the mesh size would affect the solution accuracy. Further refinement of the mesh shown in Fig. 10 yields negligible visual changes in the computed pore water pressure head field, so we could confidently incorporate the current pore water pressure field into the slope stability analysis, as described in Section 3.3. The calculated FoS is depicted in Fig. 15, and we could find many trends similar to those in Fig. 11. Specifically, FoS would generally decrease with increasing slope inclination angle α_s , and for a given α_s , FoS would be higher by incorporating the contribution of matric suction, and would continue to increase by changing the definition of χ from option 2(a) to option 2(c). In addition, the differences between the LEM and lower bound limit analysis results are higher for steeper slopes. Nevertheless, there are also some differences between Figs. 15 and 11. For a fixed parameter combination (*i.e.*, slope inclination angle α_s and suction model), this non-trivial steady-state seepage poses greater risks to slope stability than the horizontal water table of 10m in the slow drawdown scenario. Furthermore, because the negative pore

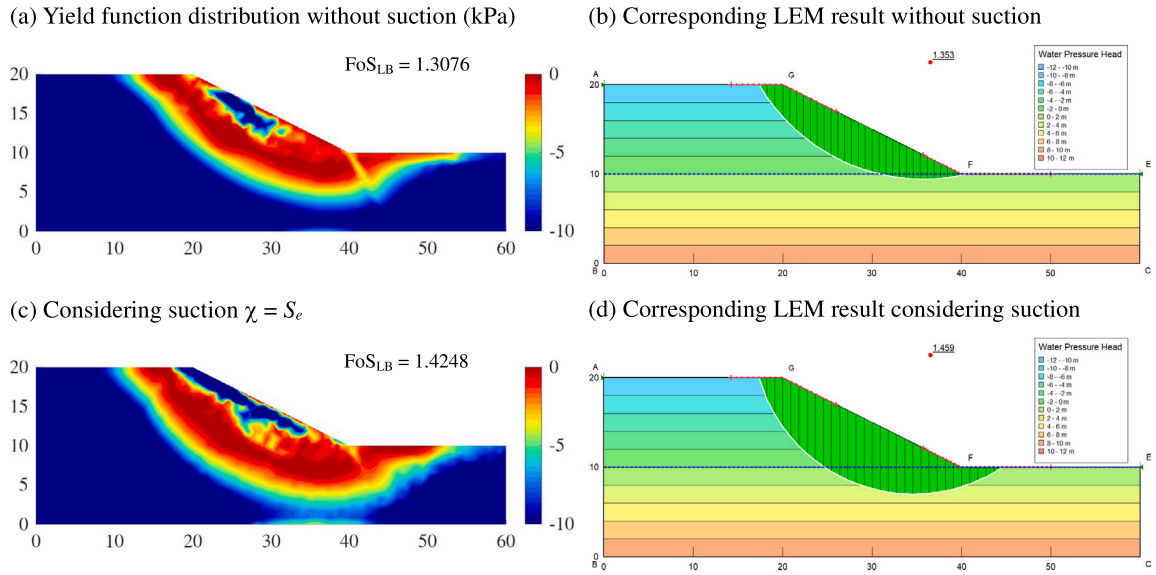


Fig. 12. Effect of the matrix suction on the failure surface pattern of a gentle slope with inclination angle $\alpha_s = \arctan(1/2)$ and horizontal water table of 10 m.

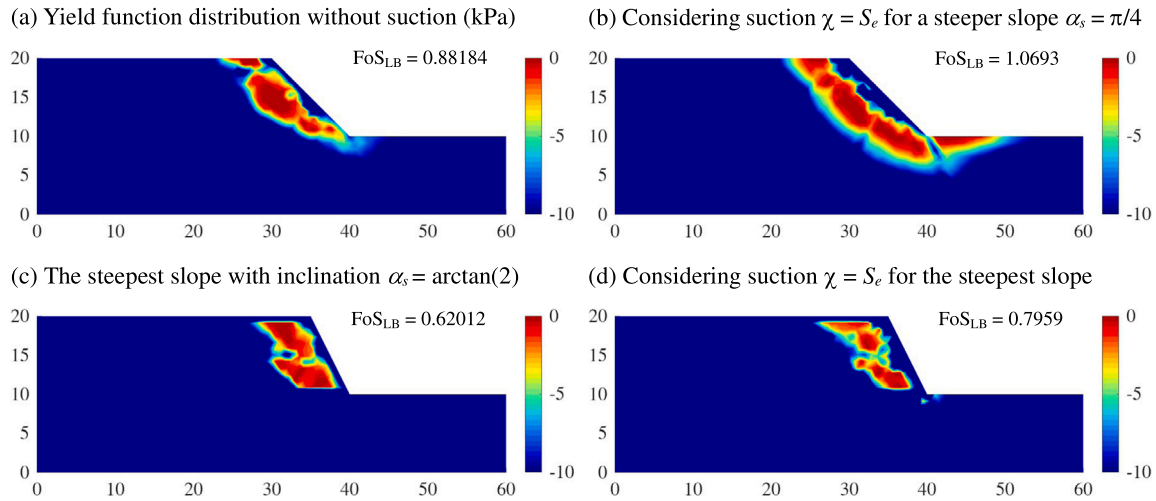


Fig. 13. Effect of the matrix suction on the failure surface pattern of two steeper slopes given a horizontal water table of 10 m.

water pressure region of Fig. 14 is smaller than that in Fig. 12(b), i.e., the scenario of slow drawdown (horizontal water table of 10m), the increase in FoS due to matrix suction in Fig. 15 is smaller than that in Fig. 11.

5.3. Rapid drawdown transient scenario

This subsection examines the rapid drawdown scenario and its impact on slope stability with a primary focus on a gentle slope characterized by an inclination angle $\alpha_s = \arctan(1/2)$. As this is a transient analysis, the time step size would affect the solution accuracy. Consequently, it is necessary to perform a time step sensitivity analysis before incorporating the pore water pressure field into the bisection algorithm of Section 3.3. Following the setup described in Section 3.1, we assume that the reservoir level is linearly decreasing from 20 m to the target elevation of 10 m in 0.01 d, and then we maintain this elevation until $t = 100$ d. Four different time-marching schemes are implemented and compared. The first two schemes uniformly divide the 100-day period into 100 and 1000 steps, corresponding to constant time steps of $\Delta t = 1$ d and $\Delta t = 0.1$ d, respectively. The third and fourth schemes divide the same 100-day period into 100 and 1000 steps, but

these are uniformly distributed on a logarithmic time scale. For the sake of brevity, these four time-marching schemes will hereafter be referred to as “100 Log steps”, “100 Linear steps”, “1000 Log steps”, and “1000 Linear steps”.

Fig. 16 depicts the temporal evolution of the total water head at Point G (located at the top corner of the slope in Fig. 2) under the four different time-marching schemes. Except for the time-marching scheme of “100 Linear steps”, the other three time-marching schemes yield nearly identical results. Similarly, Fig. 17 presents the evolution of water rate and water pressure head at six observation points under two different time-marching schemes. It can be observed that the “100 Log steps” time-marching scheme provides a sufficiently accurate solution. Furthermore, Fig. 17 provides additional valuable insight: the behavior at these observation points perfectly aligns with the potential seepage face review procedure detailed in Algorithm 1. Specifically, when the flow rate is negative (indicating outward flow), the pressure head is maintained at zero, corresponding to a Dirichlet boundary condition. Subsequently, the boundary condition is switched to a zero-flux Neumann type; consequently, the pressure head becomes negative and the soil gradually changes from saturated to unsaturated states. The transition times in Fig. 17(a) and (b) also coincide perfectly. Fig.

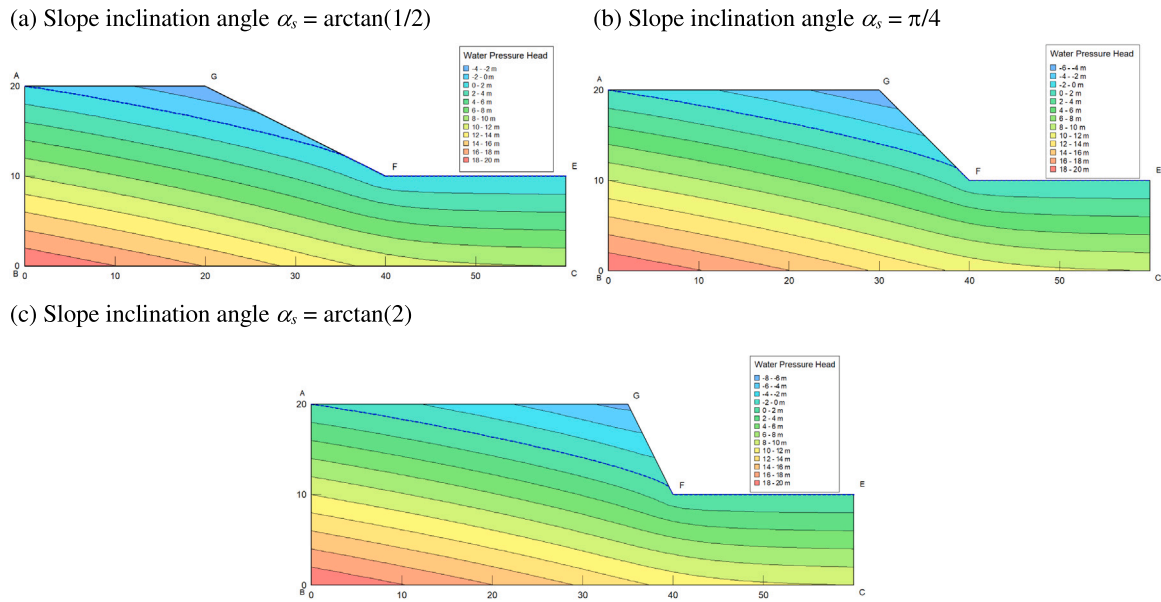


Fig. 14. Non-trivial steady-state water pressure head h distributions for three slope geometries. The thick blue dashed line indicates the contour line of zero pore water pressure (phreatic surface).

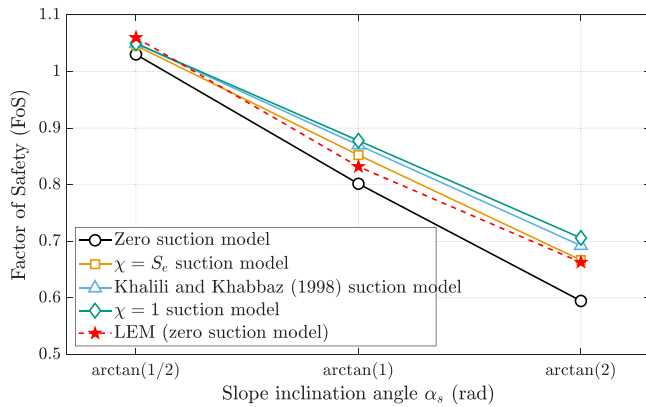


Fig. 15. FoS for different slope inclination angles and different suction models under the non-trivial steady-state seepage.

18 presents the spatial distribution of pore water pressure at four distinct time instances (0.1 d, 1 d, 10 d, and 100 d). From this figure, the progressive expansion of the unsaturated zone over time is clearly evident.

Accompanying Fig. 18 are the FoS values computed at these four time instances via lower bound finite element convex optimization. These calculations account for the contribution of the matric suction, assuming $\chi = S_e$ (option 2(a)). The results indicate that over time the FoS experiences an initial sharp decline followed by a gradual recovery. The sharp decline is observed by comparing Fig. 18(a) with Fig. 11(a), in which the water table height is 20 m with an FoS of 1.7666 (fully-saturated state). As for the gradual recovery of FoS, Eq. (4) implies that the hydraulic conductivity within the unsaturated zone is extremely low; consequently, at $t = 100$ d, the steady-state pore water pressure distribution corresponding to a slow drawdown in Fig. 12(d) has not yet been reached. Therefore, the FoS = 1.3037 remains lower than the value (1.4248) presented in Fig. 12(c). A further comparison in Tables 6 and 7 reveals that the minimum FoS under rapid drawdown is only 50% to 60% of the value obtained from the least conservative slow drawdown scenario (horizontal water table at 10 m with the suction model $\chi \equiv 1$, i.e., option 2(c)). Hence, accurately quantifying the lag of

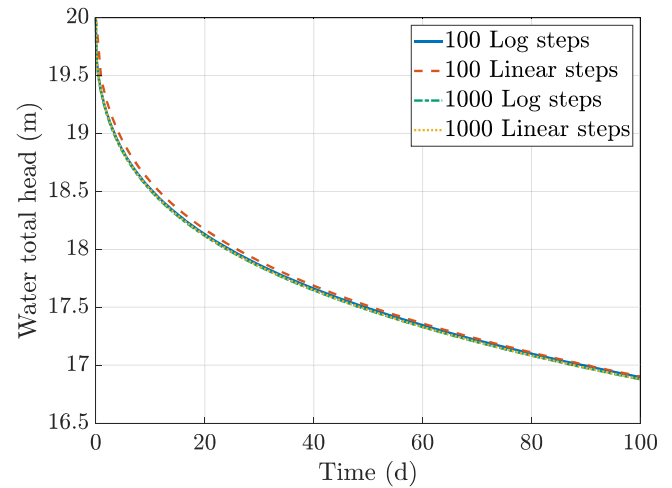


Fig. 16. Evolution of water total head at Point G (located at the top corner of the slope in Fig. 2) under four different time-marching schemes.

Table 6

Comparison of FoS between the rapid drawdown scenario (minimum FoS at $t = 0.1$ d) and the slow drawdown scenario assuming zero suction.

Slope angle	Rapid drawdown (0.1 d)	Slow drawdown (zero suction)	Ratio
$\arctan(1/2)$	0.92676	1.3076	0.71
$\pi/4$	0.65723	0.88184	0.745
$\arctan(2)$	0.4873	0.62012	0.786

the internal phreatic surface is indispensable for reliable assessment of slope stability.

5.4. Rainfall scenario

Finally, this subsection investigates the effect of rainfall on slope stability, as prolonged high-intensity rainfall is often a primary trigger for slope failure. Similar to the previous subsection, attention is focused

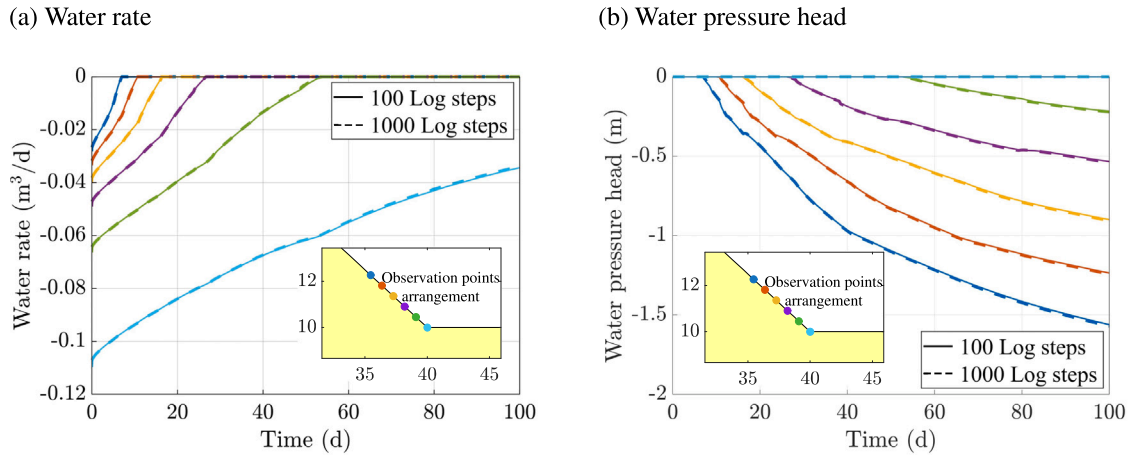


Fig. 17. Evolution of water rate and water pressure head at six observation points under two different time-marching schemes.

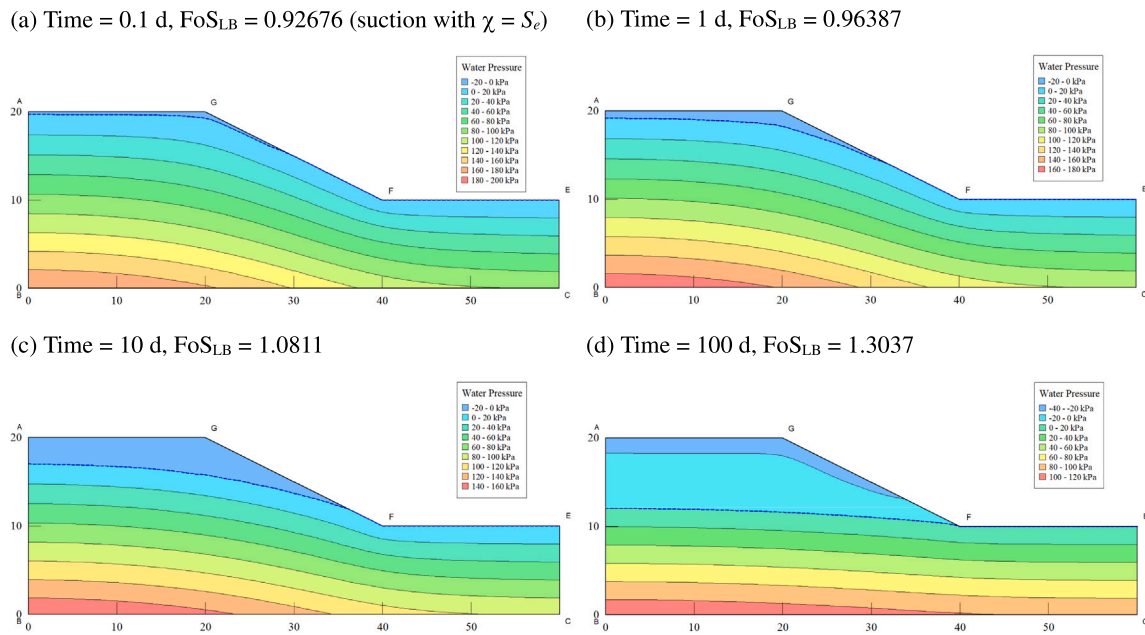


Fig. 18. Spatial distribution of pore water pressure at four distinct time instances and the associated safety factors.

Table 7

Comparison of FoS between the rapid drawdown scenario (minimum FoS at $t = 0.1$ d) and the slow drawdown scenario assuming suction model $\chi \equiv 1$, i.e., option 2(c).

Slope angle	Rapid drawdown FoS (0.1 d)	Slow drawdown ($\chi = 1$)	Ratio
$\arctan(1/2)$	0.92676	1.5205	0.61
$\pi/4$	0.65723	1.1963	0.55
$\arctan(2)$	0.4873	0.94043	0.518

on a gentle slope with an inclination angle $\alpha_s = \arctan(1/2)$. We assume that the rainfall happens at 15.2 d following the rapid reservoir drawdown described in Section 5.3, persisting for a duration of 120 h with a simulation time step of $\Delta t = 1$ hour. In the finite element unsaturated flow simulation, rainfall intensity is equivalent to a non-zero inward fluid flux across the boundary, consistent with the formulation outlined in Section 3.1. In this example, three distinct rainfall intensities are evaluated: 2 mm/h, 5 mm/h, and 10 mm/h.

Fig. 19 illustrates the evolution of FoS over the 120-hour period for three rainfall intensities. The calculation of FoS considers matric suction using option 2(a). As depicted, FoS first decreases slowly and then experiences a rapid drop, before finally stabilizing at a certain value that is independent of rainfall intensity. This final value corresponds to a saturated state given in Fig. 20(d). It is critical to distinguish this saturated state from the state in the slow drawdown scenario when water table height is 20 m. These two saturated states have different boundary fluid fluxes: the former has a non-zero boundary fluid flux due to rainfall, while the latter has an exactly zero boundary flux. The latter also has a much larger FoS (1.7666) than the former one. The whole process of water pressure evolution under a rainfall intensity of 10 mm/h is also given in Fig. 20.

6. Limitations of the model

Before the closure, a brief discussion about the limitations of the model is given here. Firstly, the hysteresis in SWCC is ignored in this study. Hysteresis can change the suction path during drainage (rapid drawdown) vs. imbibition (subsequent rainfall) and therefore

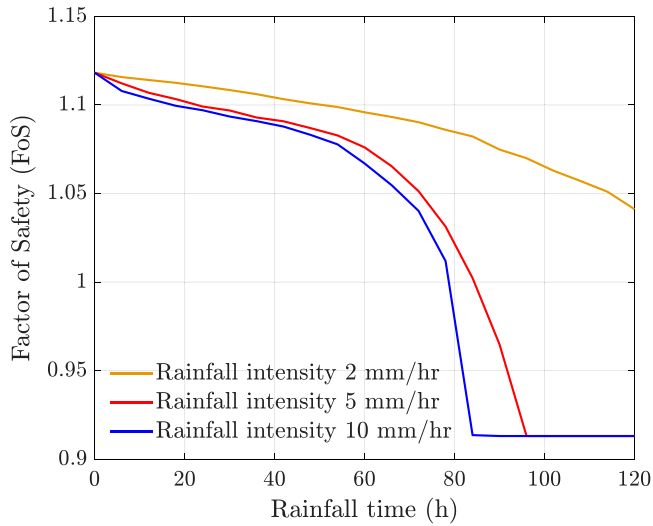


Fig. 19. Evolution of FoS under different rainfall intensities. The calculation of FoS considers matric suction using option 2(a).

change χ evolution. If hydraulic hysteresis were taken into account, the soil–water characteristic curve (SWCC), water retention capacity $C(h)$, and hydraulic conductivity $k(h)$ would differ between drainage and imbibition. Although incorporating these hysteretic effects might alter the exact values of the Factor of Safety (FoS), the overall trends and fundamental findings of our study would remain unchanged.

Secondly, the conventional single porosity medium is assumed in this work, while the possible role of preferential flow or macropores in field settings should be acknowledged (Zhang et al., 2019). With the presence of macropores, it is still possible to use the equivalent single porosity porous medium concept by calculating the equivalent SWCC and hydraulic conductivity, although this is not a trivial task for unsaturated soils. At this stage, it is challenging to predict whether preferential flow would amplify or dampen the transient effects, and we would like to leave this for future study. An alternative method is to model the soil as a double-porosity medium (Zhang et al., 2019; Yan et al., 2019) and substitute the mean pore pressure into the

limit analysis framework. However, the main challenge here is the proper definition of the Bishop coefficient for both macropores and micropores.

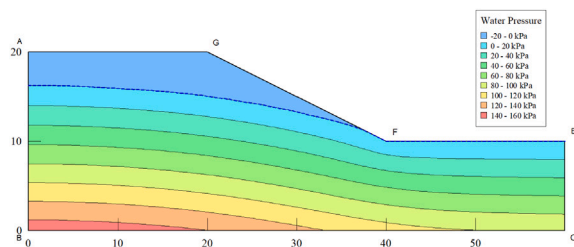
Other limitations such as different hydraulic models, different failure criteria, and heterogeneous anisotropic soil slopes (Wu et al., 2025) will also be left as future work. We believe our framework based on CVX could be “in principle” applied to these more realistic scenarios.

7. Closure

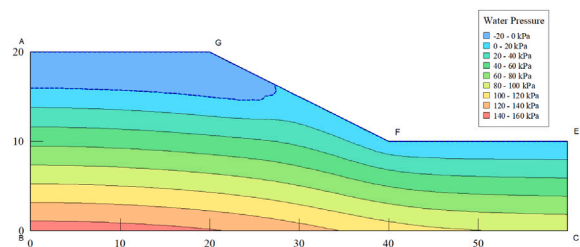
In this work, a lower bound limit analysis framework is developed to evaluate the stability of unsaturated slopes under fluctuating water levels using finite elements and convex optimization. The framework starts with the finite element discretization and potential seepage face review algorithm of the mixed-form Richards equation, from which the distribution of pore water pressure head h and effective saturation S_e within the slope could be obtained. From the results of h and S_e , the nodal χp_w and surface traction due to matric suction in the unsaturated zone could be easily obtained. Next, a novel convex formulation of finite element lower bound limit analysis of slope stability considering the nodal χp_w and surface traction due to matric suction is proposed. The convex formulation follows the DCP ruleset and includes both LP and SOCP, which could be readily solved by CVX. Through model validation, the new SOCP formulation greatly outperforms the original LP formulation, and the performance of CVX is quite promising compared with other specific solvers in literature. The model application incorporates analyses for both steady-state and transient seepage conditions, which is further divided into four typical scenarios: slow reservoir drawdown, non-trivial steady-state reservoir drawdown considering external water supply, rapid reservoir drawdown, and post-drawdown rainfall. Based on a series of parametric analyses, the following detailed conclusions are drawn:

- The impacts of different Bishop's χ formulae and matric suction are systematically quantified. Moving from the zero-suction model to χ 's definition options 2(a), 2(b), and 2(c), the FoS increases and thus becomes progressively less conservative. This suggests that option 2(a) ($\chi = S_e$) may serve as a balanced choice. The magnitude of the FoS increase highly depends on the extent of the unsaturated zone and the level of matric suction.
- During the slow reservoir drawdown, the change of FoS would generally exhibit a non-symmetric “U-shaped” curve, except for

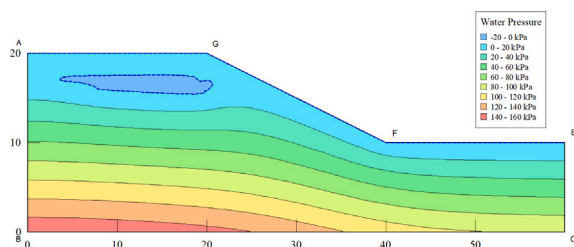
(a) Beginning of rainfall, $\text{FoS}_{\text{LB}} = 1.1182$



(b) Rainfall for 48 hours, $\text{FoS}_{\text{LB}} = 1.0831$



(c) Rainfall for 78 hours, $\text{FoS}_{\text{LB}} = 1.0118$



(d) Rainfall for 120 hours, $\text{FoS}_{\text{LB}} = 0.91309$

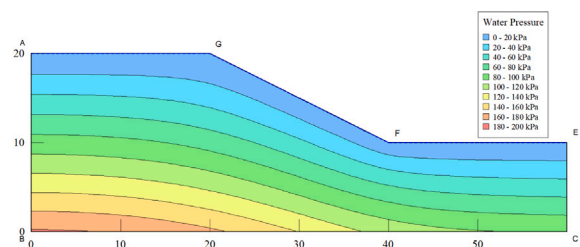


Fig. 20. Evolution of pore water pressure under a rainfall intensity of 10 mm/h. The subfigure (a) corresponds to 15.2 d after rapid drawdown, and the subfigure (d) corresponds to a saturated steady-state under rainfall.

a very steep slope, in which FoS is monotonically decreasing. For steeper slopes, the difference between our lower bound solution and LEM solution also gets larger.

- (iii) For rapid reservoir drawdown, the Factor of Safety (FoS) experiences an initial sharp decline followed by a gradual recovery. By comparing the lowest FoS value from this initial sharp decline with the FoS value under slow reservoir drawdown, a reasonable estimate for the range of the FoS under normal drawdown conditions can be obtained, as these two scenarios represent two extreme cases.
- (iv) For post-drawdown rainfall, FoS first decreases slowly and then experiences a rapid drop, before finally stabilizing at a certain value that is independent of rainfall intensity. The final value corresponds to a saturated state, which is more dangerous than another saturated state when the water table is horizontal with a height of 20 m (i.e., before any drawdown).

These results provide valuable guidance for the safety assessment of dam drawdown before a typhoon season in practice, as well as for reservoir management.

CRedit authorship contribution statement

Jianfeng Zhou: Writing – original draft, Methodology, Investigation, Formal analysis. **Qi Zhang:** Writing – review & editing, Methodology, Formal analysis, Conceptualization, Data curation, Funding acquisition, Investigation, Software, Supervision, Validation, Visualization, Writing – original draft. **Haoliang Zhang:** Data curation, Validation, Visualization, Writing – review & editing. **Xudong Kang:** Data curation, Visualization. **Zilong Zhang:** Writing – review & editing, Visualization, Validation, Methodology, Software.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. CVX MATLAB version of the optimization problem

The following code snippet corresponds to the formal mathematical Eq. (39).

Appendix B. List of symbols

B.1. Acronyms

2D/3D	Two-dimensional/Three-dimensional
CVX	MATLAB package for disciplined convex programming
DCP	Disciplined convex programming
FEM	Finite element method
FoS	Factor of safety
LAM	Limit analysis method

LBFE	Finite element lower bound limit analysis
LEM	Limit equilibrium method
LP	Linear program
SOCP	Second-order cone program
SRF	Strength reduction factor
SWCC	Soil–water characteristic curve
T3	3-node triangular element

B.2. Greek symbols

α_{vG}	Van Genuchten SWCC model parameter (1/m)
Γ_{ps}	Potential seepage boundary node set
γ	Unit weight of soil (kN/m ³)
γ_w	Unit weight of water (kN/m ³)
$\Delta \mathbf{h}^{n+1,m+1}$	Nodal increment vector of pressure head, equal to $\mathbf{h}^{n+1,m+1} - \mathbf{h}^{n+1,m}$ (m)
Δt	Time step (s)
ϵ_{tol}	Error tolerance
θ	Volumetric water content
θ_r	Residual water content
θ_s	Saturated water content
λ_F	Objective function in convex problems of Section 2.2.2, also known as the load multiplier
ρ_w	Water density (kg/m ³)
σ	Total stress tensor (Pa)
σ'	Bishop's or Terzaghi's effective stress tensor (Pa)
σ'_n	Effective normal stress (Pa)
τ	Shear stress (Pa)
ϕ'	Effective internal friction angle (deg or rad)
χ	Bishop effective stress parameter
Ω^0	Spatially-fixed control volume (m ³)

B.3. Roman symbols

A_{equil}, A^{MC}	Constraint matrices in convex problems of Section 2.2.2
b_{equil}	Constraint vectors in convex problems of Section 2.2.2 (also c^{MC} and d^{MC})
$C(h)$	Specific moisture capacity or retention water capacity (1/m)
c'	Effective cohesion (Pa)
$\mathbb{E}$	Gradient of the shape function matrix $\mathbf{N}$ (1/m)
$F_{MC}(\sigma')$	Mohr–Coulomb yield function (Pa ²)
$\tilde{F}_{MC}(\sigma')$	Equivalent convex form of $F_{MC}(\sigma')$ (Pa)
$\mathbf{g}$	Gravity acceleration vector (m/s ²)
$\mathbf{H}_{F_{MC}}$	Hessian matrix of $F_{MC}(\sigma')$
$\mathbf{h}$	Nodal pressure head vector in FEM (m)
h	Hydraulic pressure head (m)
$\mathcal{K}^{n+1,m}$	Tangential operator matrix (m ² /s)
$k(h)$	Unsaturated hydraulic conductivity (m/s)
k_{sat}	Saturated hydraulic conductivity (m/s)
m_{vG}	Van Genuchten SWCC model parameter
$\mathbf{N}$	Shape function matrix of the hydraulic pressure head h
$\mathbf{n}$	Face normal vector in Section 2.2.2
$\mathbf{n}$	Outward-pointing unit normal vector defined on boundary
$\mathbf{n}_g$	Unit gravity vector
n_{elm}	Number of triangular elements
n_{face}	Number of faces in the T3 mesh

```

A_sqrt_opt = [1, -1, 0;
              0, 0, 2];
A_trace = [1, 1, 0];

cvx_solver SDPT3
cvx_begin quiet
    % declare stress vector and scalar variables, num_variable = num_T3elem * 9
    variables X_stress(num_variable) lambda_F(1);
    maximize lambda_F % overload coefficient
    subject to
        % face traction continuity constraints
        % hypothetical loading + matric suction contributions (fixed for latter)
        Aequil_d*X_stress == lambda_F*bequil_d + bequil_d_suction;

        % stress equilibrium constraints
        % separate contributions from p_w and gravity
        Aequil_e*X_stress == bequil_e_h + bequil_e_g;

        % following lines are for the MC yield constraints
        Transformed_Stress = A_sqrt_opt * reshape(X_stress, 3, []);
        Transformed_trace = A_trace * reshape(X_stress, 3, []);
        norms(Transformed_Stress, 2, 1) ...
        <= 2 * coh_reduced * cosd(phi_deg_reduced) ...
        - Transformed_trace*sind(phi_deg_reduced); % hyperbolic cone
cvx_end

```

n_{node}	Number of nodes in the T3 mesh (in standard FEM and lower bound FEM, the values are different, for the latter, $n_{\text{node}} = 3 \times n_{\text{elm}}$)
n_{vG}	Van Genuchten SWCC model parameter
$P(\mathbf{n})$	Traction projection matrix associated with $\mathbf{n}$ in Section 2.2.2
p	Number of sides used in the linearized yield polygon of the original Mohr–Coulomb yield surface, see Fig. 1(d)
p_a	Air pressure (Pa)
p_w	Pore water pressure (Pa)
q_w	Darcy seepage velocity of water (m/s)
$\bar{q}_h$	Boundary inward flux (m/s)
$\mathcal{R}^{n+1,m}$	Residual vector (m ³ /s)
$r(\mathbf{n})$	Shear projection row vector associated with $\mathbf{n}$ in Section 2.2.2
S_e	Effective saturation
s	Matric suction stress, and it is equal to $p_a - p_w$ (Pa)
s_e	Suction value parameter in the model of Khalili and Khabbaz (1998) (Pa)
t'	Traction vector (Pa)
t	Time (s)
X_σ	Variable to be optimized in convex problems (Pa)
$\mathbf{x}$	Position vector in physical domain (m)
x	Horizontal coordinate in 2D, same as the first component of $\mathbf{x}$ (m)
y	Vertical coordinate in 2D, same as the second component of $\mathbf{x}$ (m)

B.4. Miscellaneous symbols

$\mathbf{I}$	Second-order identity tensor
$1_{\{\circ\}}$	Indicator function

$\mathcal{S}$	Mass-specific source/sink term (1/s)
$\langle \circ \rangle$	Macaulay bracket notation
$(\circ)_{ss}$	To emphasize the steady-state condition
$\ \circ\ $	Vector L2 norm
$(\circ)^{n+1}$	Time step level $n + 1$
$(\circ)^{n+1,m}$	Iteration counter m in the current time step
∇	Gradient operator (1/m)
$\mathcal{X}_\sigma^k$	Global stress quantity obtained from X_σ in Section 2.2.3 (Pa)

Data availability

Data will be made available on request.

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