



Enhanced THM coupling for anisotropic geomaterials and smoothed-FEM simulation

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ABSTRACT

As the thermo-hydro-mechanical (THM) coupling phenomenon becomes more and more common in various engineering applications, a deeper understanding of the behavior of the porous media under combined “THM loading” would be extremely useful. As researchers in computational geomechanics, we first revisit the established poromechanics theory and then move one step forward to (1) derive a generalized fluid flow equation that incorporates both mechanical and thermal strain anisotropy; and to (2) propose an analogous poro-elasto-plastic structural heating/cooling term that bridges not only to the phenomenological porothermoelasticity but also to the uncoupled standard thermal equation. Next in the section of the numerical method, we enrich the stabilized node-based smoothed finite element method (SNS-FEM) to incorporate THM coupling, which leads to the development of smoothed nodal thermal strain and smoothed thermal strain gradient. Subsequently, the proposed numerical framework is verified against three benchmark tests, which aim for the correct implementation of heat convection and THM contact, as well as the comparison with FEM. Finally, two hypothetical model application examples are presented, which investigate the effect of mechanical anisotropy and cross-coupling in THM biaxial test and the effect of temperature in pipeline penetration, respectively. Overall, this study offers a promising contribution to the understanding and modeling of THM problems.

1. Introduction

Thermo-hydro-mechanical (THM) coupling is prevalent in various engineering applications [1], including conventional oil and gas production [2], production from hydrate-bearing sediments [3], nuclear waste treatment [4], enhanced geothermal system [5,6], hydraulic fracturing [7–9], and so on. In the submarine engineering, we always have the submarine pipeline laid on the seabed or buried into the soil that is used for the transportation of oil and gas [10–12]. However, under unfavorable circumstances, the pipeline could suffer from thermally-induced lateral buckling or uplifting [13], which could ultimately lead to catastrophic damage. However, current research on submarine pipelines often overlooks the thermal effects on the surrounding soil [14,15]. These aforementioned engineering activities necessitate an in-depth understanding of the THM coupling.

In the framework of continuum mechanics, nonlinear partial differential equations governing the multi-coupled interactions of thermal, hydraulic, and mechanical fields have been developed and solved numerically for initial boundary value problems. For thermo-mechanical coupling, Prof. Christian Miehe and his co-workers have made exceptional scientific achievements in terms of formulation and numerical implementation at finite strain [16,17], gradient thermoplasticity [18], thermoviscoplasticity [19], and phase field modeling of fracture in multi-physics problems [20,21]. As for THM coupling in porous media, Noorishad et al. [22] and Aboustit et al. [23] presented the earliest efforts by extending Biot's theory of consolidation and presented the corresponding finite element THM scheme. The continued interest in coupled THM phenomena has encouraged the development of many

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computer codes on THM coupling [24], and in the year of 2001, Rutqvist et al. [25] summarized the governing equations used in several popular computer codes, from which we could observe the differences in mathematical derivations to these coupled equations. Recently, researchers have extended the THM coupling to thermo-hydro-mechanical freezing (THM-F) coupling [26,27], which includes a novel thermo-mechanical decomposition treatment of the solid phase [26], and others combine THM coupling with uncertainty analysis on reservoir parameters and reservoir modes [28], or with discrete fractures [29,30], solute transport [31], double porosity [32], multiscale modeling [33], and phase field model or peridynamics model of hydraulic fracturing [34–36]. As a result, more sophisticated multi-physics simulation platform such as OpenGeoSys has been released and is currently widely used in science and engineering [37]. Theoretically speaking, the mathematical basis for the THM coupling could be divided into two main categories: phenomenological theory [38] and mixture theory [39–42], and for the latter category, a non-exhaustive list of representative works could date back to the theory of Bowen [43], theory of Lewis and Schrefler [44–46], theory of Borja [39,47,48], Theory of Porous Media (TPM) by Ehlers [41], Poromechanics by Coussy [40], and the Averaging Theory (TCAT) by Gray [49]. Despite the abundance in mathematical theory, a key question observed from various literature remains: What is the correct and consistent simplification of the energy conservation equation that could lead to a form ready for numerical discretization? *i.e.*, the final form only depends on the primary unknowns (displacement, pore pressure, and temperature) and we could easily find its connection with the thermally uncoupled counterpart, which is widely used in current THM literature [24,28,37,38,50]. In addition, the effect of mechanical anisotropy on the THM governing equations has not been clearly elucidated using mixture theory. In this work, we aim to address the above-mentioned knowledge gap.

Over the years, various numerical methods have been developed for THM problems, including the finite element method (FEM) [51] and its variations such as X-FEM [52], hybrid XFEM-ECM [30,53], enriched-FEM [54], and multiscale FEM [55,56], the material point method (MPM) [57], the discrete element method (DEM) [58], and the finite discrete element method (FDEM) [9]. In terms of these methods, FEM is known for its solid mathematical foundation and accuracy. It has been applied to various structural and geotechnical challenges, including identifying localized failure zones in composite domes [59], conducting free vibration and buckling analysis of Timoshenko beams with complex cross-sections, material anisotropy, and biaxial load [60], and performing stability analysis of wellbores in gas-bearing unsaturated formations [61]. While the traditional standard FEM stores the information of stress and other history variables at Gauss points, which would lose accuracy during adaptive remeshing, the node-based strain smoothing technique is believed to overcome this drawback by transforming the storage location of variables from integration points to nodes [62–65]. This method is known as NS-FEM in literature [62,64], and in addition, NS-FEM tends to give a “softer” solution that could alleviate the overly stiff problem of FEM with linear triangular element [62,66]. However, this “softer” solution could be problematic as well, resulting in temporary instability issues [67,68]. Several remedies have been proposed to address this temporal instability, categorized into three primary groups: (i) The α -FEM and hybrid smoothed FEM (H-SFEM) [69,70]; (ii) Least-square residual stabilization, which adjusts the potential energy function by introducing a squared-residual term [68,71]; (iii) Gradient-based stabilization, which aims to recover the strain gradient within the smoothing domain [72–74]. A noteworthy advantage of the gradient-based stabilization method is the elimination of the artificial parameter (α) found in the other two methods. The stabilized NS-FEM, abbreviated as SNS-FEM, which achieves close-to-exact stiffness while at the same time maintaining nodal integration characteristics, has recently attracted growing interest. For example, Long et al. [75] combined S-FEM with SPH for Fluid-Structure Interaction (FSI) problems; Ren et al. [65] used the SNS-RPIM to analyze the behavior of functionally graded magneto-

electro-elastic structures; and Wu et al. [76] extended the S-FEM to multi-material, multi-component biomechanical analysis.

However, the treatment of thermal strain hinders the development of SNS-FEM in addressing THM issues. The main reason is that there is no spatial gradient in the expression of thermal strain, which cannot be “smoothed” in the same way as that for total strain. Currently, there are two ways to handle this inconsistency. The first method calculates the thermal strain and treats it as the initial strain [77]. However, this approach is limited to addressing thermoelastic steady-state problems. The second method considers the effect of thermal strain as an external load [78], but it is only applicable to elastic problems as the fourth-order elastoplastic tangent modulus tensor is in general not a constant. Both methods have significant shortcomings and can be considered as uncoupled thermal analysis methods. In this study, the concept of smoothed thermal strain is proposed, which merges the calculation methods of total strain and thermal strain within the framework of SNS-FEM. This enables the transient elastoplastic THM coupling analysis.

With these key issues in mind, this work presents an improved thermo-hydro-mechanical coupling theory, and based on which we develop an SNS-FEM THM coupling simulator. Therefore, the results of this paper are useful in shedding light onto the physical meaning and rigorous derivation of the governing conservation equations, in particular, how to incorporate anisotropy and how to derive an analogous term $\mathcal{K}$ firstly proposed by Simo and Miehe [16]. By ignoring this term $\mathcal{K}$ and dissipation, the temperature evolution equation downgrades to its standard form used in many textbooks. Numerical examples are provided for elucidating the role of mechanical anisotropy and cross-coupling term through the THM biaxial test, as well as the role of temperature on pipeline penetration resistance. The paper is organized as follows. Section 2 presents the theoretical development on THM coupling. Section 3 presents the detailed extension of SNS-FEM into THM coupling. Section 4 provides a comprehensive verification set for the proposed numerical scheme. Numerical application examples are then provided in Section 5, followed by a conclusion.

2. Governing equations for anisotropic thermo-poro-elasto-plasticity

In this section, we present the balance principle (*i.e.*, mass, momentum, and energy) for ideal porous media under thermo-poro-elasto-plasticity. Before the specification of the balance equations, the following assumptions should be first clarified.

1. The porous material is idealized as a continuum (binary) mixture that consists of two constituents, the solid and fluid, where the solid constituent forms the solid skeleton, and the fluid constituent fully occupies the pores inside the solid skeleton, *i.e.*, a fully saturated situation.
2. Based on the mixture theory, if the representative elementary volume exists, then each constituent occupies a fraction of volume at the same macroscopic material point [39,79].
3. There is no restriction on the compressibility of solid constituent.² This is in contrast to most literature on soil mechanics, which assumes the solid constituent to be incompressible.
4. The solid skeleton is not restricted to the case of isotropy, that is to say, the governing equations have taken anisotropy into consideration.
5. The fluid flow follows linear Darcy’s law and we assume the fluid constituent to be slightly compressible or incompressible.
6. There is no mass transfer between solid and fluid constituents.

² In literature, solid constituent is sometimes also known as solid grain, or without causing any ambiguity, people would simply use the term “solid” [80]. However, the solid constituent and solid skeleton have different meanings and should be distinguished with each other.

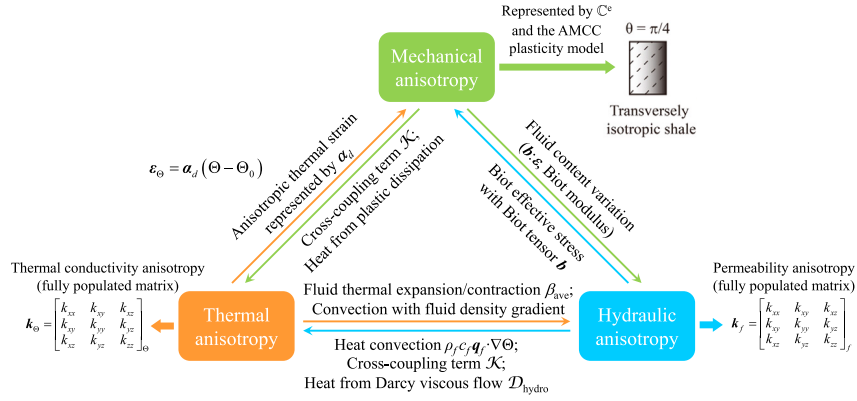


Fig. 1. The schematic representation for the improved thermo-hydro-mechanical coupling in anisotropic elastoplastic porous media. While all the symbols will be explained in the following text, we can clearly see that anisotropy affects every single field as well as the two-way coupling between any two fields. The blue and green arrows pointing to the orange thermal block are not always clearly elucidated in previous studies.

- There is instantaneous local thermal equilibrium between the solid constituent and the fluid constituent. That is to say, only one temperature field is needed in the derivation.
- The thermal expansion of solid skeleton is not necessarily equal to the thermal expansion of solid constituent. In particular, the thermal expansion of solid skeleton could even be anisotropic, particularly in the context of thermal pressurization [81–83].

By comparing with classical thermo-hydro-mechanics theory as reported in the previous studies, the main difference on model assumptions lies on the explicit consideration of anisotropy, as depicted in Fig. 1, which would lead to the generalized Biot coefficient tensor, generalized Biot modulus, and generalized bulk average volumetric thermal expansion coefficient. In addition, we show a correct and consistent simplification of the energy conservation equation, which leads to a familiar thermal equation form with an improved poro-elasto-plastic structural heating/cooling term $\mathcal{K}$. Consequently, the energy consumption due to cross-coupling, elastoplastic deformation, fluid diffusion, heat convection, and outward heat flux are all formulated in the current modeling framework, which is also shown in Fig. 1.

For the sake of brevity, in the following derivations, we would mainly highlight the salient features relevant to the thermal effects. For the case of hydro-mechanical coupling in anisotropic porous media, interested readers may refer to the work by Zhao and Borja [84,85], Zhang and Borja [86], and Zhang et al. [87,88].

2.1. Mass and momentum conservation

We begin with the mass conservation equation. Since the thermal effect would not alter the conservation of mass along the motion of any control volume dV , the following two equations would still hold (cf. Zhang and Borja [86])

$$\frac{1-\phi}{\rho_s} \frac{d\rho_s}{dt} - \frac{d\phi}{dt} + (1-\phi)\nabla \cdot \mathbf{v} = 0, \quad (1)$$

$$\frac{d\phi}{dt} + \frac{\phi}{\rho_f} \frac{d\rho_f}{dt} + \frac{\nabla \rho_f}{\rho_f} \cdot \mathbf{q}_f + \nabla \cdot \mathbf{q}_f + \phi \nabla \cdot \mathbf{v} = 0, \quad (2)$$

where Eqs. (1) and (2) describe the mass balance for solid constituent and fluid constituent, respectively. In the above two equations, ϕ represents the (Eulerian) porosity, ρ_s is the intrinsic density of solid, ρ_f is the intrinsic density of fluid, $\mathbf{v}$ represents the velocity field of the solid skeleton, and $\mathbf{q}_f$ is known as the Darcy velocity that represents the “average” relative motion between the solid and fluid across face dA of control volume dV [89].

Nevertheless, Eqs. (1) and (2) are not the finalized forms, and in Zhao and Borja [84] and Zhang and Borja [86], authors adopted the barotropic law to continue the derivation. The barotropic law says that

the density is a univariate function of pressure. However, when we consider thermal coupling, the barotropic law is not valid, and it is replaced by the following three equations

$$\frac{1}{\rho_s} \frac{d\rho_s}{dt} = \frac{1}{K_s} \frac{dp_s}{dt} - \beta_s \frac{d\Theta}{dt}, \quad (3)$$

$$\frac{1}{\rho_f} \frac{d\rho_f}{dt} = \frac{1}{K_f} \frac{dp}{dt} - \beta_f \frac{d\Theta}{dt}, \quad (4)$$

$$\frac{\nabla \rho_f}{\rho_f} = \frac{\nabla p}{K_f} - \beta_f \nabla \Theta. \quad (5)$$

In other words, the density is a multivariate function of both pressure and temperature. In the above three equations, p_s and K_s represent the intrinsic pressure and bulk modulus of the solid, $p = p_f$ and K_f are the fluid pressure and fluid bulk modulus, Θ represents the single temperature field, and β_s and β_f represent the volumetric thermal expansion coefficient, which differ from the linear thermal expansion coefficient by a factor of three [50]. It is also worthwhile to mention that, in general, p_s is not necessarily equal to p , except for the case of the unjacketed test [38].

Another key difference is that the total strain tensor $\boldsymbol{\varepsilon}$ is now expressed as the summation of elastic strain $\boldsymbol{\varepsilon}^e$, plastic strain $\boldsymbol{\varepsilon}^p$, and thermal strain $\boldsymbol{\varepsilon}_\Theta$. For thermal strain, we assume a linear relationship with the temperature variation denoted by the linear thermal strain coefficient tensor $\boldsymbol{\alpha}_d$, as shown by the following equation

$$\boldsymbol{\varepsilon}_\Theta = \boldsymbol{\alpha}_d (\Theta - \Theta_0), \quad (6)$$

where Θ_0 denotes the initial or reference temperature. We can see from Eq. (6) that it considers the anisotropic thermal expansion of the solid skeleton [81–83]. During the simplification process, we also need the form of the effective stress principle. According to Noorishad et al. [22], any rise or fall of temperature acts in the form of initial strains that are independent of other mechanical stresses, therefore, the effective stress principle takes the following form in THM coupling, which is also supported by many publications such as Huang and Ghassemi [90], Shi and Durucan [91], and Salimzadeh et al. [92]

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p \mathbf{b} = \mathbb{C}^e : \boldsymbol{\varepsilon}' = \mathbb{C}^e : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p - \boldsymbol{\varepsilon}_\Theta), \quad (7)$$

where $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}$ are the effective stress and total stress tensors, respectively, $\mathbb{C}^e$ is the fourth-order elastic stiffness tensor, $\mathbf{b}$ is the Biot coefficient tensor and it is calculated as

$$\mathbf{b} = \mathbf{1} - \frac{\mathbb{C}^e : \mathbf{1}}{3K_s}, \quad (8)$$

where $\mathbf{1}$ is the second-order identity tensor.

Following similar simplification strategies of Zhang and Borja [86] and by making use of Eqs. (6) and (7), the generalized fluid flow

equation under THM coupling and anisotropy reads

$$\mathbf{b} : \frac{d\epsilon}{dt} + (1 - \mathbf{b}) : \frac{d\epsilon^p}{dt} - \beta_{\text{ave}} \frac{d\Theta}{dt} + \frac{1}{M} \frac{dp}{dt} + \frac{\nabla \rho_f}{\rho_f} \cdot \mathbf{q}_f + \nabla \cdot \mathbf{q}_f = 0, \quad (9)$$

where

$$\beta_{\text{ave}} = (1 - \phi)\beta_s + \phi\beta_f - (1 - \mathbf{b}) : \alpha_d \quad (10)$$

is the generalized bulk average volumetric thermal expansion coefficient, M is the generalized Biot modulus [38,84,86], which can be expressed as

$$\frac{1}{M} = \frac{\phi}{K_f} + \frac{1}{K_s} \left(\frac{\mathbf{b} : \mathbf{1}}{3} - \phi \right), \quad (11)$$

and $(\nabla \rho_f)/\rho_f$ is evaluated using Eq. (5). At this point, Eq. (9) is theoretically solvable after inserting Darcy's law for $\mathbf{q}_f$ with parameters $\mathbf{k}_f$ and γ_f , which are the hydraulic conductivity coefficient tensor and specific weight of the fluid, respectively.

By referring to Eq. (10) and the assumption 8, this study provided the most general form of β_{ave} , and we can easily see that this form is compatible with other published results, such as the result obtained by Cheng [38]. To be more specific, the consistency is achieved by assuming an isotropic porous medium and by considering that the thermal expansion coefficients of the solid skeleton and the solid constituent are equal.

We would like to close this subsection by addressing the momentum conservation equation. Similar to other published literature, we assume that the thermal effect would not create any new momentum supply. Therefore, by ignoring the inertia forces, the balance of linear momentum takes the following form in the current configuration [79]

$$\nabla \cdot \boldsymbol{\sigma} + [\rho_f \phi + \rho_s(1 - \phi)]\mathbf{g} = \mathbf{0}, \quad (12)$$

where the second term represents the body force of the binary mixture induced by gravity.

2.2. Thermodynamics

Our starting point is the energy conservation and dissipation inequality (first and second laws of thermodynamics) expressed in the reference configuration [40], i.e., Lagrangian form, which reads

$$\frac{d\hat{e}}{dt} + J \nabla \cdot (e_f \rho_f \mathbf{q}_f) - \boldsymbol{\tau} : \mathbf{d}^s - (J \rho_f \mathbf{g}) \cdot \mathbf{q}_f + J \nabla \cdot (p \mathbf{q}_f) + J \nabla \cdot \mathbf{q}_\Theta - \hat{r}_\Theta = 0, \quad (13)$$

$$\hat{D}_{\text{int}} = \frac{d\hat{\eta}}{dt} + J \nabla \cdot (\eta_f \rho_f \mathbf{q}_f) + J \nabla \cdot \left(\frac{\mathbf{q}_\Theta}{\Theta} \right) - \frac{\hat{r}_\Theta}{\Theta} \geq 0, \quad (14)$$

where the hat symbol ($\hat{\cdot}$) is adopted to represent quantity per unit reference volume Ω_0 (Lagrangian quantity). Therefore, $\hat{e}$ and $\hat{\eta}$ are the overall internal energy density and entropy density per unit Ω_0 , respectively; e_f and η_f are the fluid-specific (per mass unit) internal energy and entropy, respectively; $J = \det(\mathbf{F})$ is the deformation Jacobian of the deformation gradient tensor $\mathbf{F}$; $\mathbf{q}_\Theta$ and $\hat{r}_\Theta$ denote the heat flux vector and heat source, respectively; $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ is the Kirchhoff stress tensor of the solid skeleton, and $\mathbf{d}^s$ represents the rate of deformation tensor of the solid skeleton.

However, Eqs. (13) and (14) cannot be used directly and they are far from our familiar form. Therefore, in the following paragraphs, we move one step forward to give a more familiar form of the temperature equation, which would be comparable with those in published literature.

Now we define the overall Helmholtz free energy density function $\hat{\psi}$ as $\hat{\psi} = \hat{e} - \Theta \hat{\eta}$, together with the decomposition of $\hat{\psi}$ and $\hat{\eta}$ as

$$\hat{\psi} = \hat{\psi}_s + \rho_f \hat{\phi} \psi_f; \quad \hat{\eta} = \hat{\eta}_s + \rho_f \hat{\phi} \eta_f, \quad (15)$$

where $\hat{\psi}_s$ and $\hat{\eta}_s$ represent the free energy density and entropy density of the solid skeleton, respectively, and $\hat{\phi} = J\phi$ is known as the Lagrangian porosity. By substituting Eq. (15) into Eqs. (13) and (14) and

after several steps of algebraic simplifications, we obtain the so-called reduced dissipation inequality as

$$\Theta \hat{D}_{\text{int}} = \underbrace{\left\{ \boldsymbol{\sigma} : \frac{d\epsilon}{dt} + p \frac{d\hat{\phi}}{dt} - \hat{\eta}_s \frac{d\Theta}{dt} - \frac{d\hat{\psi}_s}{dt} \right\}}_{D_{\text{mech}}} + \underbrace{\left\{ \mathbf{q}_f \cdot (-\nabla p + \rho_f \mathbf{g}) \right\}}_{D_{\text{hydro}}} + \underbrace{\left\{ \frac{\mathbf{q}_\Theta}{\Theta} \cdot (-\nabla \Theta) \right\}}_{D_{\text{th}}} \geq 0. \quad (16)$$

We can see that the dissipation comes from three parts as represented by three curly brackets: the mechanical part D_{mech} , the hydraulic part D_{hydro} , and the thermal part D_{th} . For the thermal part, the term D_{th} implies the Fourier's law; and for the hydraulic part, the multiplication in D_{hydro} leads to the Darcy's law. These linear constitutive laws ensure the dissipations D_{th} and D_{hydro} to be non-negative with positive semi-definite coefficient tensors. Lastly, for the mechanical part, by assuming a certain form of $\hat{\psi}_s$ as [33]

$$\hat{\psi}_s = \hat{\psi}_s(\epsilon - \epsilon^p, \hat{\phi} - \hat{\phi}^p, \Theta, \boldsymbol{\zeta}), \quad (17)$$

where $\hat{\phi}^p$ is the plastic (Lagrangian) porosity [93] and $\boldsymbol{\zeta}$ is a vector of strain-like plastic internal variables analogous to the scalar cumulative plastic shear strain, the standard Coleman–Noll arguments would lead to the following state equations [18,39,84]

$$\boldsymbol{\sigma} = \frac{\partial \hat{\psi}_s}{\partial (\epsilon - \epsilon^p)}; \quad p = \frac{\partial \hat{\psi}_s}{\partial (\hat{\phi} - \hat{\phi}^p)}; \quad \hat{\eta}_s = -\frac{\partial \hat{\psi}_s}{\partial \Theta}, \quad (18)$$

and we are left with the reduced plastic dissipation as

$$D_{\text{mech}}^p = \boldsymbol{\sigma} : \frac{d\epsilon^p}{dt} + p \frac{d\hat{\phi}^p}{dt} + \left(-\frac{\partial \hat{\psi}_s}{\partial \boldsymbol{\zeta}} \right) \circ \frac{d\boldsymbol{\zeta}}{dt} \geq 0, \quad (19)$$

where $\boldsymbol{\kappa} \triangleq -\partial \hat{\psi}_s / \partial \boldsymbol{\zeta}$ is a stress-like vector of plastic internal variables characterizing the hardening response. The reduced plastic dissipation Eq. (19) suggests the form of yield criterion that depends on $\boldsymbol{\sigma}$, p , and $\boldsymbol{\kappa}$.

Remark 2.1. To ensure the derivation correctness, we have made a quick comparison with the finite strain coupled thermoplasticity theory proposed in Simo and Miehe [16] by setting $p = 0$ in Eqs. (18) and (19), and the corresponding equations from the reference work [16] are copied here (notations are quite standard in finite deformation multiplicative plasticity, thus their meanings are not repeated here)

$$\boldsymbol{\tau} = 2 \frac{\partial \hat{\psi}_s}{\partial \mathbf{b}^e} \mathbf{b}^e; \quad \hat{\eta}_s = \eta^p - \frac{\partial \hat{\psi}_s}{\partial \Theta}, \quad (20)$$

$$D_{\text{mech}}^p = \boldsymbol{\tau} : \left[-\frac{1}{2} (\mathcal{L}_v \mathbf{b}^e) \cdot (\mathbf{b}^e)^{-1} \right] + \left(-\frac{\partial \hat{\psi}_s}{\partial \alpha} \frac{d\alpha}{dt} \right) + \Theta \frac{d\eta^p}{dt} \geq 0. \quad (21)$$

It can be seen by comparing with Eqs. (18) and (19) that the results are completely analogous except that Simo and Miehe [16] introduced the plastic configurational entropy η^p which is related to dissipative plastic structural changes and does not affect the internal energy, as a result, the yield function explicitly contains the temperature field in that work.

We next use the reduced dissipation to simplify the energy conservation Eq. (13), the result is shown below

$$\Theta \left(\frac{d\hat{\eta}_s}{dt} + \rho_f \hat{\phi} \frac{d\eta_f}{dt} + \rho_f \mathbf{q}_f \cdot \nabla \eta_f \right) = D_{\text{mech}}^p + D_{\text{hydro}} - \nabla \cdot \mathbf{q}_\Theta + r_\Theta. \quad (22)$$

Through a Legendre transform $\hat{G}_s = \hat{\psi}_s - p(\hat{\phi} - \hat{\phi}^p)$, which introduces the Gibbs free energy function $\hat{G}_s = \hat{G}_s(\epsilon - \epsilon^p, p, \Theta, \boldsymbol{\zeta})$, Eq. (18) becomes

$$\boldsymbol{\sigma} = \frac{\partial \hat{G}_s}{\partial (\epsilon - \epsilon^p)}; \quad \hat{\phi} - \hat{\phi}^p = -\frac{\partial \hat{G}_s}{\partial p}; \quad \hat{\eta}_s = -\frac{\partial \hat{G}_s}{\partial \Theta}. \quad (23)$$

By substituting the last sub-equation of Eq. (23) and the state equation for fluid entropy η_f [40] into Eq. (22), the final form of the temperature evolution equation emerges as

$$C \frac{d\Theta}{dt} + \rho_f c_f \mathbf{q}_f \cdot \nabla \Theta = \left[D_{\text{mech}}^p + D_{\text{hydro}} - \mathcal{K} \right] - \nabla \cdot \mathbf{q}_\Theta + r_\Theta, \quad (24)$$

where c_f is the fluid-specific heat capacity,

$$C \triangleq -\Theta \frac{\partial^2 \hat{G}_s}{\partial \Theta^2} + \rho_f \phi c_f \quad (25)$$

is defined as the overall specific heat (at constant deformation, fluid pressure, and internal variables) per unit volume, and

$$\begin{aligned} \mathcal{K} \triangleq -\Theta \left[\frac{\partial^2 \hat{G}_s}{\partial \Theta \partial (\epsilon - \epsilon^p)} : \frac{d(\epsilon - \epsilon^p)}{dt} + \frac{\partial^2 \hat{G}_s}{\partial \Theta \partial \zeta} \circ \frac{d\zeta}{dt} + \frac{\partial^2 \hat{G}_s}{\partial \Theta \partial p} \frac{dp}{dt} \right. \\ \left. + \beta_f \phi \frac{d^f p}{dt} \right] \quad (26) \end{aligned}$$

is named **poro-elasto-plastic structural heating/cooling** (analogous to Simo and Miehe [16]), which is related to cross-coupling. This term $\mathcal{K}$ is not explicitly mentioned in previous THM literature and as shown below, by specifying the expression of $\hat{G}_s$, we could reach the consistency with the porothermoelasticity theory of Cheng [38]. In addition, ignoring dissipation and $\mathcal{K}$ terms in Eq. (24) would lead to the thermally uncoupled form, which is still widely used in current THM literature [57,94].

Remark 2.2. In this work, we adopt the following expression for $\hat{G}_s$

$$\begin{aligned} \hat{G}_s = & -\phi_0 p - \eta_s^0 \Theta \\ & + \frac{1}{2} (\epsilon - \epsilon^p) : \mathbb{C}^e : (\epsilon - \epsilon^p) - p \mathbf{b} : (\epsilon - \epsilon^p) \\ & - (\epsilon - \epsilon^p) : \mathbb{C}^e : \alpha_d (\Theta - \Theta_0) \\ & + \beta_{sk} p (\Theta - \Theta_0) - \frac{p^2}{2N} - \frac{C_{sk} (\Theta - \Theta_0)^2}{2\Theta_0} + \underbrace{K^p(\zeta)}_{\text{hardening}}, \quad (27) \end{aligned}$$

where the script “0” implies the initial state, Θ_0 is a reference absolute temperature in $^\circ\text{K}$, and we assume an initially stress-free and zero fluid pressure condition in the above $\hat{G}_s$. Under small perturbations of temperature, C and $\mathcal{K}$ can be approximated as

$$C \approx C_{sk} + \rho_f \phi c_f; \quad \mathcal{K} \approx \Theta_0 \left[\alpha_d : \mathbb{C}^e : \frac{d(\epsilon - \epsilon^p)}{dt} - \beta_{sk} \frac{dp}{dt} - \beta_f \phi \frac{d^f p}{dt} \right]. \quad (28)$$

Under the case of porothermoelasticity, the plastic strain tensor ϵ^p drops out. If we further assume that α_d is isotropic, we would find an exact one-to-one correspondence between the coefficients in porothermoelasticity (Eq. (11.91) of Cheng [38]) and the coefficients in Eq. (28). This one-to-one correspondence suggests the correctness of our derivation in the incorporation of anisotropy into the energy balance equation.

Furthermore, we could conduct another theoretical verification of the anisotropic energy balance equation by comparing Eq. (27) with Eq. (4.20) in Page 75 of Coussy's Poromechanics book under the case of elasticity. When $\mathbb{C}^e$ is isotropic and characterized by the bulk modulus K and shear modulus μ , $\mathbf{b}$ and α_d are isotropic and characterized by scalars b and α , respectively, and β_{sk} is equal to $3\alpha_\phi$ if $\beta_s = 3\alpha$, the above two equations would coincide with each other completely. In fact, even under elastoplasticity, the term $\epsilon - \epsilon^p$ is still justified by Shao et al. [33].

Remark 2.3. In Eq. (27), the parameters N and β_{sk} are not specified yet using material properties. By comparing the state equation for $\hat{\phi} - \hat{\phi}^p$ in Eq. (23) and the generalized fluid flow equation Eq. (9), we obtain

$$\frac{1}{N} = \frac{1}{K_s} \left(\frac{\mathbf{b} : \mathbf{1}}{3} - \phi \right); \quad \beta_{sk} = (1 - \phi) \beta_s - (1 - b) : \alpha_d. \quad (29)$$

At this time, the Gibbs free energy function $\hat{G}_s$ is fully specified using measurable material properties, which implies Eq. (24) is theoretically solvable and ready for numerical discretization. The numerical treatment will be given in the next section.

It is worthwhile to mention that the expression of β_{sk} , or equivalently β_{ave} , still adds some new insights into the literature. In the

expression, β_s is derived from the change of the intrinsic solid density ρ_s , while α_d is derived from the thermal deformation of the bulk volume (or porous medium frame). There is no guarantee that we should have $\beta_s = \mathbf{1} : \alpha_d \triangleq \beta_d$ unless we adopt the ideal porous medium assumption [38], see Eq. (11.43) and Eq. (11.52) of Cheng [38]. In addition, β_e in Eq. (11.67) of Cheng [38] also contains three independent parameters β_s , β_d , and β_f after inserting Eq. (11.60), and our β_{ave} is a generalized version of β_e accounting for anisotropy. Similarly, in Cui et al. [95], the authors also mentioned that $\beta_s = \beta_d$ holds only under certain conditions.

Remark 2.4. In Chapter 11.4.4 of Cheng [38], the author proposed a dimensionless thermoelastic coupling parameter, such that when this parameter is much less than one, the thermoelastic coupling would be weak and we could use the thermally uncoupled form. In this work, we extend this parameter to include anisotropy and denote it as ϖ_Θ

$$\varpi_\Theta = \frac{\bar{K} \text{Tr}^2(\alpha_d) \Theta_0}{C} \ll 1, \quad (30)$$

where $\bar{K}$ is the generalized drained bulk modulus defined as [96]

$$\bar{K} = \frac{\mathbf{1} : \mathbb{C}^e : \mathbf{1}}{9}. \quad (31)$$

3. Numerical discretization by SNS-FEM

For the sake of brevity, we omit the details of strong form and weak form, and directly give the matrix residual form of Eqs. (9), (12) and (24) as

$$\mathcal{R}_u = \int_B \mathbf{B}^\top (\boldsymbol{\sigma}' - p \mathbf{b}) dV - \mathbf{F}_{EXT} = \int_B \mathbf{B}^\top (\boldsymbol{\sigma}'' - p \mathbf{1}) dV - \mathbf{F}_{EXT}, \quad (32)$$

$$\begin{aligned} \mathcal{R}_p = \int_B \mathbf{N}_p^\top \left[\frac{p - p_n}{M} - \beta (\Theta - \Theta_n) + \mathbf{1} : (\epsilon - \epsilon_n) \right. \\ \left. + (b - \mathbf{1}) : (\epsilon^e - \epsilon_n^e) \right] dV \quad (33) \end{aligned}$$

$$\begin{aligned} + \Delta t \int_B \mathbf{N}_p^\top \left(\frac{\nabla \rho_f}{\rho_f} \cdot \mathbf{q}_f \right) dV - \Delta t \int_B \mathbf{E}_p^\top \mathbf{q}_f dV - \mathbf{G}_{EXT}, \\ \mathcal{R}_\Theta = \int_B \mathbf{N}_\Theta^\top C (\Theta - \Theta_n) dV + \Delta t \int_B \mathbf{N}_\Theta^\top \left(\rho_f c_f \mathbf{q}_f \cdot \nabla \Theta - \mathcal{D}_{hydro} \right) dV \\ - \int_B \mathbf{N}_\Theta^\top \left[\boldsymbol{\sigma} : (\epsilon^p - \epsilon_n^p) + p (\hat{\phi}^p - \hat{\phi}_n^p) + \kappa \circ (\zeta - \zeta_n) \right] dV \\ + \int_B \mathbf{N}_\Theta^\top \Theta_0 \left[\alpha_d : \mathbb{C}^e : (\epsilon - \epsilon_n - \epsilon^p + \epsilon_n^p) - \beta_{ave} (p - p_n) \right] dV \\ - \Delta t \int_B \mathbf{E}_\Theta^\top \mathbf{q}_\Theta dV - \Delta t \int_B \mathbf{N}_\Theta^\top r_\Theta dV - \mathbf{H}_{EXT}, \quad (34) \end{aligned}$$

where quantity $(\circ)$ at the previous time step t_n is denoted as $(\circ)_n$, while quantities without any subscript are evaluated at the current time step t_{n+1} . In the above three equations, $\mathbf{N}_u$, $\mathbf{N}_p$, and $\mathbf{N}_\Theta$ stand for the shape functions of displacement, fluid pressure, and temperature, respectively; $\mathbf{B}$, $\mathbf{E}_p$, and $\mathbf{E}_\Theta$ represent the spatial derivatives of $\mathbf{N}_u$, $\mathbf{N}_p$, and $\mathbf{N}_\Theta$, respectively; $\boldsymbol{\sigma}''$ is the well-known Terzaghi effective stress, which is largely responsible for the plastic deformation and yield criterion [84]; $\beta = (1 - \phi) \beta_s + \phi \beta_f$ is an intermediate thermal expansion parameter; $\mathbf{F}_{EXT}$ is the external nodal force vector induced by the body force and surface traction, and $\mathbf{G}_{EXT}$ and $\mathbf{H}_{EXT}$ are external nodal fluid flux and heat flux vectors, respectively. These residual arrays are generally nonlinear with respect to the unknown vectors, and are best solved using Newton's method, which requires linearization [84]. The derived results are presented in Appendix A.

In NS-FEM, we construct the smoothing domain from the T3 mesh, as shown by Fig. 2. The irregular smoothing domain of a node k is created among the surrounding elements by connecting their centroids and edge midpoints sequentially. As a result, all the residuals in Eqs. (32) to (34) are assembled from the smoothing domains rather than from the triangular elements. In SNS-FEM, the irregular smoothing domain in NS-FEM can be approximated by the red circle with the

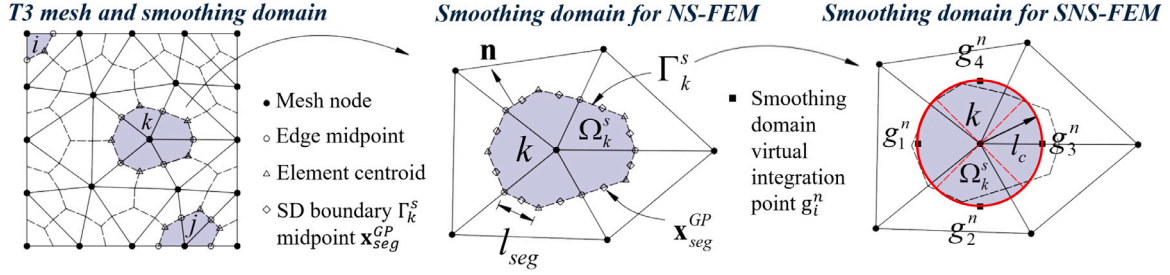


Fig. 2. Schematic of the construction of the smoothing domain for NS-FEM and SNS-FEM. We start by connecting the element centroids and the edge midpoints surrounding node k sequentially, and then approximate the irregular polygon smoothing domain with a circle according to the area equivalence principle.

same area and four virtual integration points. Thus, the strain, stress, and plastic internal variables (state variables) are all stored at the four virtual integration points. Here, we would only highlight the calculation (smoothing) process for the thermal strain ϵ_θ , since the smoothing process for the compatible strain ϵ in SNS-FEM is already well-established, and we just need to realize that in thermo-mechanical or THM coupled plane strain problems, $\epsilon_z \equiv 0$ but $(\epsilon_\theta)_z$ is usually non-zero. Similarly, the smoothing process of the pressure gradient could also be found in Wang et al. [66] and Zhang et al. [87], which leads to the construction of $\bar{\mathbb{E}}_p$, and since we adopt the same linear interpolation functions for p and θ fields, the smoothed $\bar{\mathbb{E}}_\theta$ is thus equal to the smoothed $\bar{\mathbb{E}}_p$.

During the calculation, we first apply the smoothing technique to get the smoothed nodal thermal strain $\bar{\epsilon}_\theta^k$ as

$$\bar{\epsilon}_\theta^k = \frac{1}{A_k^s} \int_{\Omega_k^s} \epsilon_\theta(\mathbf{x}) d\Omega = \bar{\mathbb{N}}_\theta^k \boldsymbol{\theta}, \quad (35)$$

where

$$\bar{\mathbb{N}}_\theta^k = \frac{1}{A_k^s} \int_{\Omega_k^s} \left\{ \alpha_d \right\}_{6 \times 1} \underbrace{\begin{bmatrix} N_1 & \dots & N_J & \dots & N_P \end{bmatrix}}_{\mathbb{N}_\theta^k}^{(\theta)} d\Omega, \quad (36)$$

$\boldsymbol{\theta}$ is the nodal temperature vector, and P is the number of support nodes of the smoothing domain Ω_k^s with area A_k^s . The above calculation is known as nodal integration over the smoothing domain. However, different from the gradient smoothing of the total strain tensor ϵ in NS-FEM, the divergence theorem cannot be applied here as there is no gradient operation in Eq. (6), so this calculation cannot be converted into a boundary integral. In Li et al. [97], $\bar{\epsilon}_\theta^k$ is called artificial smoothed thermal strain. Next, to obtain the smoothed thermal strain at four virtual integration points, we consider a first-order Taylor expansion of the thermal strain, as shown by the following equation

$$\bar{\epsilon}_\theta = \bar{\epsilon}_\theta^k + \bar{\epsilon}_{\theta,x}^k (x - x_k) + \bar{\epsilon}_{\theta,y}^k (y - y_k). \quad (37)$$

In other words, through Taylor expansion, we try to recover the thermal strain variation within the smoothing domain, which reflects the essence of SNS-FEM. Now we need to evaluate the smoothed thermal strain gradient, and this time, we have $\nabla \bar{\epsilon}_\theta$, which means $\bar{\epsilon}_{\theta,x}^k$ and $\bar{\epsilon}_{\theta,y}^k$ are calculated by integrating the $\mathbb{N}_\theta$ on the boundary of the smoothing domain Γ_k^s , as shown below

$$\begin{aligned} \bar{\epsilon}_{\theta,i}^k &= \frac{1}{A_k^s} \int_{\Omega_k^s} \frac{\partial \epsilon_\theta(\mathbf{x})}{\partial i} d\Omega = \left\{ \frac{1}{A_k^s} \int_{\Omega_k^s} \left\{ \alpha_d \right\}_{6 \times 1} \frac{\partial \mathbb{N}_\theta^k}{\partial i} d\Omega \right\} \boldsymbol{\theta} \\ &= \left\{ \frac{1}{A_k^s} \int_{\Gamma_k^s} \left\{ \alpha_d \right\}_{6 \times 1} \mathbb{N}_\theta^k n_i d\Gamma \right\} \boldsymbol{\theta} = \bar{\mathbb{N}}_{\theta,i}^k \boldsymbol{\theta} \quad (i = x, y), \end{aligned} \quad (38)$$

and one may find that this operation is equivalent to the calculation of smoothed $\mathbb{B}$ matrix in standard NS-FEM for solid mechanics [64].

Finally, as we have already calculated the $\bar{\epsilon}_\theta^k$ and $\bar{\epsilon}_{\theta,i}^k$ ($i = x, y$), we could evaluate $\bar{\epsilon}_\theta$ at those four integration points using Eq. (37) where $l_c = \sqrt{A_k^s/\pi}$. This is the additional step in SNS-FEM introduced by thermal coupling.

Remark 3.1. In SNS-FEM, the combination of elements with different interpolation orders as that in the Taylor–Hood scheme is undesirable, as we want to use simplicial linear element to reach a better accuracy and lower computational cost. One drawback of the elements with equal-order interpolation is that they do not satisfy the inf-sup condition and numerical solutions of the pressure field would exhibit oscillation in the undrained limit [98,99]. Various stabilization methods have been proposed for equal-order elements, including the finite increment calculus (FIC) method [100], Galerkin/least-squares formulation (GLS) based on mass conservation equation [101], the polynomial pressure projection (PPP) method [102,103], the bubble function method [104], the fluid pressure Laplacian (FPL) method [101], and so on. Among these methods, the PPP method stands out for the absence of high-order shape function construction and formulation simplicity. The fundamental concept behind the PPP method revolves around introducing a stabilization term into the weak form of the governing equations to satisfy a weak inf-sup condition. In this work, the PPP approach only applies to the fluid flow equation, in which the stabilizing terms take the same form as those in White and Borja [102], and they are scaled by the same stabilization parameter C_p , which is derived by Sun [105].

Remark 3.2. It is also useful to clarify the comparison of SNS-FEM with gradient-based regularization techniques in standard FEM. To the best of the authors' knowledge, these two methods are completely different, as they are designed to circumvent different problems. The smoothing approach employed here, the node-based strain smoothing technique in NS-FEM, is specifically designed to mitigate the over-stiffness of standard FEM with linear triangular elements. This innovative approach not only alleviates the aforementioned over-stiffness but also strategically transforms the storage location of variables from integration points to nodes. In fact, this smoothing approach in NS-FEM is somehow similar to the reduced integration or one-point Gauss integration in standard FEM, which may lead to rank-deficiency and hour-glass mode. The gradient-based stabilization method in SNS-FEM is then developed to improve the performance of NS-FEM. That is to say, in standard FEM, we do not need such gradient-based stabilization as long as we use full integration for stiffness matrix.

As for the standard gradient-based regularization techniques employed in FEM, it is introduced to handle the pathologic mesh dependence when there is strain-softening [106]. When strain softening occurs under solid dynamics, domain splits into an elliptic part with imaginary wave speed (standing wave) and hyperbolic part where waves can propagate. Due to the inability of the standing wave to propagate, the localization zone has zero thickness with no energy consumption, which contradicts with experimental evidence. Therefore, we need to introduce an internal length scale to circumvent the pathologic mesh dependence and give the localization zone a finite thickness.

To summarize, without any regularization of governing equations, the smoothing approach employed here would also display pathologic mesh dependence when there is strain-softening, as there is still no internal length scale in our method.

Remark 3.3. We want to add the final remark on the virtual integration points, especially its accuracy during remeshing. The gradient-based stabilization method introduces virtual integration points, which are associated with the nodes but are not actual points within the mesh. After remeshing, the smoothing domain is changed (perhaps both shape and area), but all the information in these virtual integration points is preserved and enter the next step. Throughout the entire nonlinear mechanical analyses, all variables are stored at the virtual integration points of the nodes, and the related calculations are also performed at these points. The averaging operation is not performed during the nonlinear analyses; it is only executed after the computation has concluded (*i.e.*, during post-processing) for plotting contour maps of historical variables. Even if remesh occurs, these nodes and their corresponding virtual integration points will be retained. Therefore, the accuracy is not affected. Additionally, when remeshing is performed, the stresses at the virtual integration points of the newly added nodes are obtained through direct interpolation from the stresses at the existing virtual integration points. To summarize, this averaged nodal stress does not participate in the actual mechanical analysis and, therefore, has a minimal impact on the accuracy of the nonlinear mechanical analysis.

4. Multi-physical verification

In this section, we provide several benchmark examples to verify the capability of our proposed numerical scheme in handling multi-physics coupling. All the benchmark examples contain the thermal field, and gravity has explicitly been neglected in the calculations. All the finite element meshes in this section and the next section are based on the linear triangle element, and for every node in the mesh, it contains four degrees of freedom (DoF): 2 for displacement, 1 for pore pressure, and 1 for temperature. Therefore, once the total node number is given, we could easily calculate the total DoFs.

4.1. Heating of a permeable material through the moving liquid

The first benchmark example is designed to assess the hydro-thermal coupling effect, which involves the heating process of a permeable material through the movement of the liquid. As depicted in Fig. 3, a rectangular region occupied by a permeable material has a length of $L = 2$ m and a height of $H = 1.5$ m, which is discretized into 2471 nodes and 4800 triangle elements. In this study, we assume the following material parameters: the porosity is $\phi = 0.1$, the intrinsic permeability is 10^{-11} m², the fluid viscosity is $\mu_f = 0.001$ Pa·s, the fluid density is $\rho_f = 1000$ kg/m³, the fluid-specific heat capacity is $c_f = 1100$ J/kg/K, the overall specific heat is $C = 560$ kJ/m³/K, and the overall thermal conductivity coefficient is $k_\theta = 1.85$ W/m/K.

In the designed scenario, the porous material experiences a pressure of 20 kPa on the left boundary, while maintaining a constant zero pressure on the right boundary. As a result, the steady-state flow of pore liquid is established along the horizontal direction. The steady-state flow would carry the high temperature from the left boundary (marked by the trapezoid) and heat the whole material from 0°C. In our simulation, the maximum time is 7000 s and the time increment per step is $\Delta t = 50$ s. Although we use the implicit scheme, we still calculate the Courant–Friedrichs–Lewy (CFL) number as

$$\text{CFL} = \frac{u_x \Delta t}{\Delta x} = \frac{10^{-4} \times 50}{0.025} = 0.2 < 1, \quad (39)$$

where $u_x = 10^{-4}$ m/s is the steady-state fluid flow velocity and $\Delta x = 0.025$ m is the mesh size (smallest height of an element). Both the specific left boundary temperature and the analytical temperature distribution are given in Chapter 2.6.2 of Kolditz et al. [37] and will not be repeated here.

Fig. 4 illustrates the heating process by using the contour plot of temperature. In contrast, if we ignore the convective heat transfer and

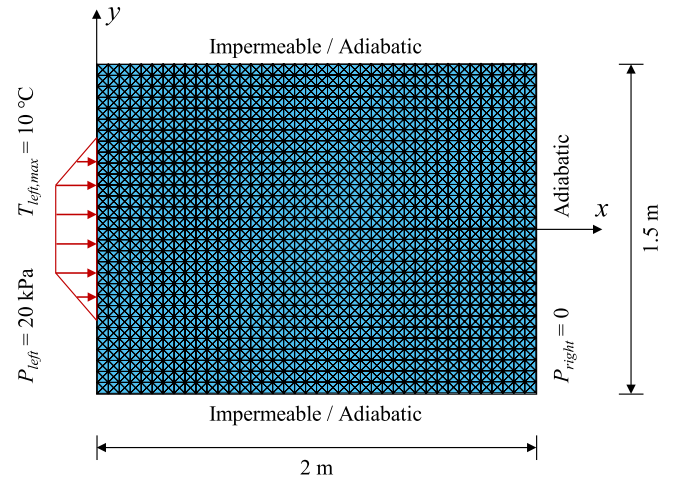


Fig. 3. Geometry and mesh of the model. Due to the heat convection, the steady-state flow would carry the high temperature from the left boundary (marked by the trapezoid) and heat the whole material. Boundary conditions have been clearly marked in the figure. Through this example, we could verify the correctness in the implementation of the heat convection module.

only consider heat conduction, we find that the heating process is significantly delayed. In literature, the relative intensity of heat convection to heat conduction can be described using the Péclet number Pe as

$$Pe = \frac{u_x L_x}{\kappa_\theta} = 60.54 \gg 1, \quad (40)$$

where $L_x = 2$ m is the model length in the horizontal direction, and $\kappa_\theta = k_\theta / C$ is the heat diffusivity coefficient. We can see from the value of Pe that convection dominates heat transfer.

Finally, we perform a sensitivity analysis on the time step size and a convergence study on both the mesh size and the time step size. The results are summarized in Fig. 5. In (a) and (b), the relative temperature error is calculated as

$$E_\theta = \sqrt{\frac{\int_\Omega (\theta^{\text{num}} - \theta^{\text{ana}})^2 d\Omega}{\int_\Omega (\theta^{\text{ana}})^2 d\Omega}}, \quad (41)$$

and we find that the convergence rate on mesh size matches well with the theoretical value of FEM, which means that the error from time discretization is not yet dominant. From (c) and (d), we can see at a 10-s interval, the difference from the analytical solution is almost imperceptible to the naked eye. The logarithmic error decreases linearly with the linear time step size, as shown by (b), where the error from spatial discretization is not yet dominant as well.

As an aside, we want to discuss the issues of discretization and modeling errors by taking advantage of this example. Discretization errors in numerical analysis typically arise from the approximation of continuous equations through temporal and spatial discretization schemes. These errors depend on the resolution of the computational grid and the time step size; coarser grids or larger time steps can lead to reduced accuracy. Additionally, the degree of the shape function (linear or quadratic) also affects discretization errors. To determine these errors, we can rely on the analytical solution, as shown in Eq. (41). For cases without analytical solutions, we may treat the simulation results obtained using the finest mesh (reaching the maximum capability of the computer) as the true solution. A relevant example is given in Appendix B. To mitigate these errors, grid refinement studies (h-refinement and r-refinement), p-refinement (using higher-order shape functions), and time step convergence tests are often employed to ensure that the numerical results are independent of the discretization parameters within acceptable tolerances.

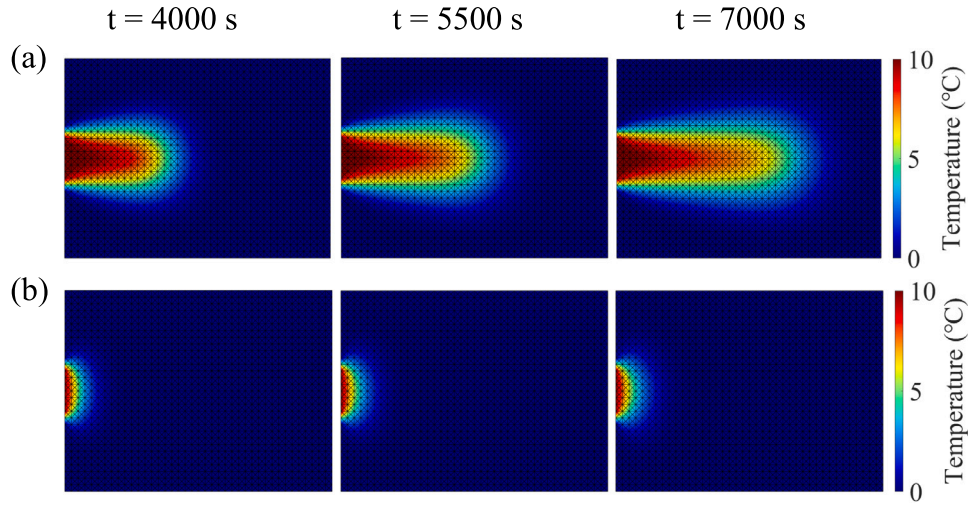


Fig. 4. Temperature distributions for the case of (a) with convection and (b) without convection. As we can clearly see the gradual heating in the first row, but it is negligible in the second row, which suggests that in this example, the heat conduction effect is much weaker than the heat convection.

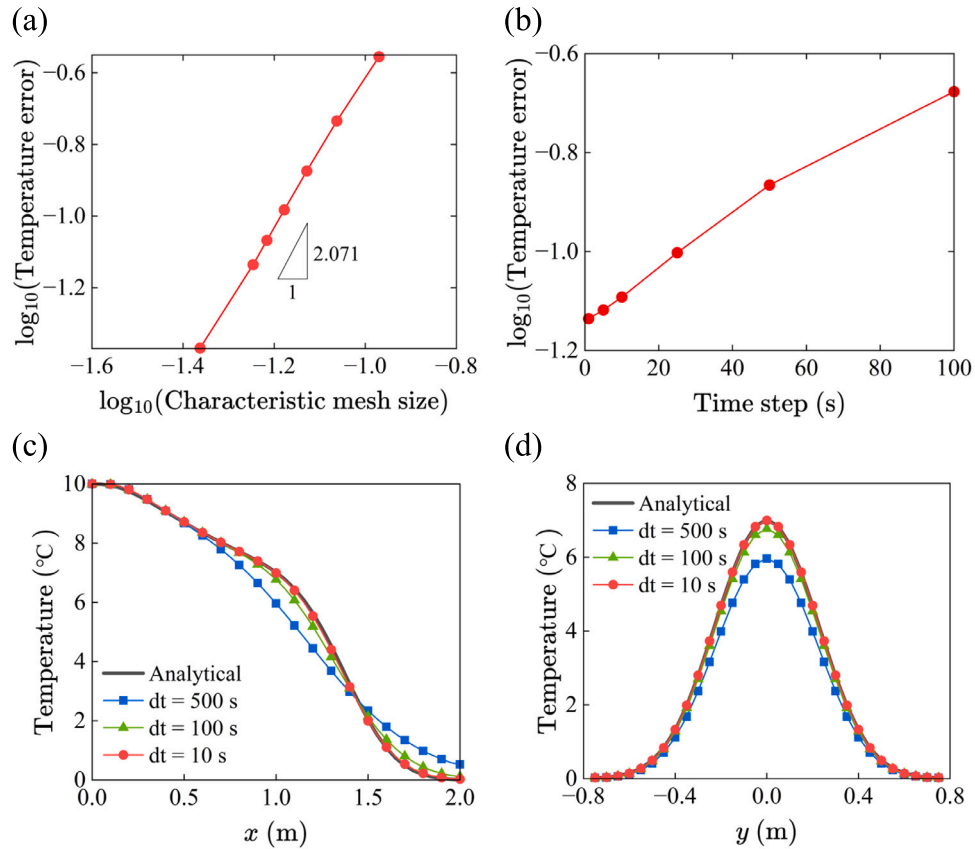


Fig. 5. Hydro-thermal coupling example: (a) temperature error versus different mesh densities while the time step is fixed at 1 s, the slope matches well with that of FEM; (b) temperature error versus different time step sizes while the mesh from Fig. 3 is used; comparison with analytical solutions for temperature distributions along (c) $y = 0$ and (d) $x = 1$ m using three different time step sizes.

On the other hand, modeling errors stem from assumptions and simplifications made in the mathematical representation of physical phenomena. These include the use of constitutive laws that may not fully capture material behavior, simplifications of complex boundary conditions, and the omission of spatial heterogeneity in material properties. Determination and mitigation of these errors require careful validation of the model against experimental data, sensitivity analyses, and, where applicable, upscaling techniques to account for spatial heterogeneity. By systematically identifying and quantifying these sources

of error, the reliability of numerical predictions can be significantly improved.

4.2. Heating of a half space

The second verification example examines the implementation of the THM coupling. The geometric configuration and mesh details are depicted in Fig. 6(a), in which two pieces of non-matching grids with

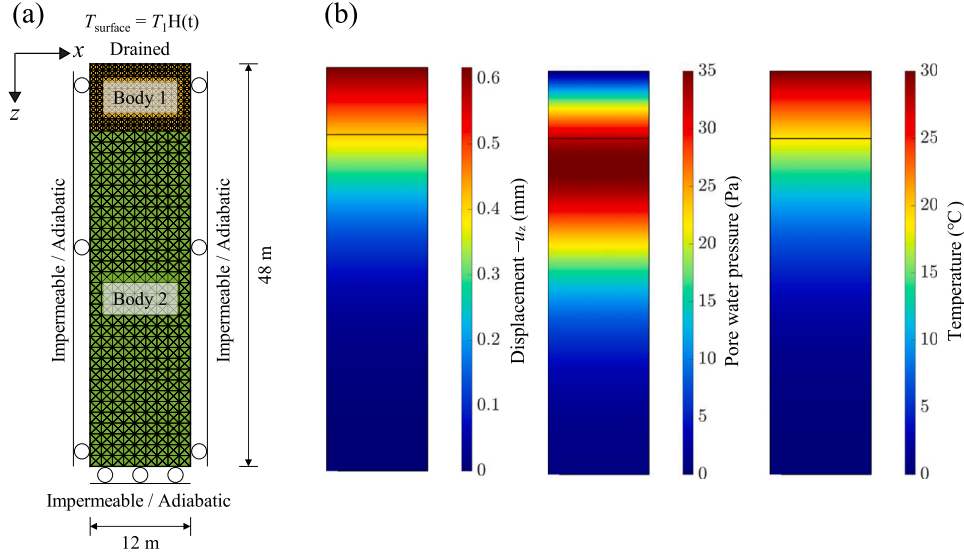


Fig. 6. Heating of a half space: (a) geometry and mesh (two pieces of non-matching grids) of the soil column that approximates the half space; (b) vertical displacement, excess pore water pressure, and temperature distribution at $t = 12$ h ($R_q = 0.5$).

Table 1

Material properties of the soil column shown in Fig. 6(a). The value of k_f is back-calculated after prescribing the value of R_q . According to the value of ω_θ , the thermally uncoupled form can be used here.

Parameter	Value	Unit
Reference temperature θ_0	273.15	K
Instantaneous temperature rise T_1	30	°C
Thermal conductivity coefficient k_θ	0.836	kW/m/K
Overall specific heat per unit volume C	167.2	kJ/m ³ /K
Fluid-specific heat capacity c_f	167.2	J/kg/K
Thermal expansion coefficient β_{ave}	1.98×10^{-6}	1/K
β_f from Eq. (4) for fluid	6.3×10^{-6}	1/K
Linear thermal strain coefficient α_d	3×10^{-7}	1/K
Young's modulus E	1	MPa
Poisson's ratio ν	0.4	–
Porosity ϕ	0.2	–
Biot coefficient	1	–
Fluid bulk modulus K_f	$+\infty$	GPa
Fluid specific weight γ_f	9.8	kN/m ³
Hydraulic conductivity coefficient k_f	T.B.D. from R_q	m/s
Dimensionless R_q	0.1, 0.25, 0.5	–
Dimensionless coupling parameter ω_θ	2.2×10^{-9}	–

tied contact are adopted. The Body 1 has 380 nodes and 704 triangle elements, and the Body 2 has 599 nodes and 1116 triangle elements. As a result, we use the Lagrange multiplier method to solve the contact problem [107–111]. Note here besides the standard mechanical contact, the contact surface also adds constraints on the pressure, temperature, fluid flux, and heat flux continuity, so we name it as the multi-physics contact, though a little bit ambitious. In our setup, the top boundary is traction-free and fully drained, while the remaining boundaries are roller-type constrained, impermeable, and adiabatic. Here, we choose a relatively large model depth to approximate the half-space. At $t = 0^+$, a temperature increase T_1 is instantaneously applied to the top boundary and maintained at a constant level thereafter. Material properties are given in Table 1. In this problem, we adopt a dimensionless number R_q to describe the system response, as suggested by Cheng [38]. R_q is the square root of the ratio of the consolidation coefficient to the heat diffusivity coefficient. Details about the calculation formula and analytical solutions are provided in Appendix C.

There are two types of time settings used in this benchmark example: (i) For cases with varying R_q , the model maximum time is calculated as $t_{\max} = (H/n)^2 / (4\xi_m c_v)$ (for example, if $R_q = 0.5$, then $t_{\max} = 800$ s), where $H = 48$ m is the height or depth of the domain,

$\xi_m = 4$ is an artificially chosen number, and $n = 12$ is a scaling factor to prevent the bottom boundary from being affected by seepage or heat transfer; (ii) For the case with a constant $R_q = 0.5$, we investigate the sensitivity on the time step size, therefore, we have three different values of $t_{\max}$: 1 min, 1 h, and 12 h. The number of time steps for all simulations is consistently 100, with a uniform time step size.

Fig. 6(b) depicts the contours in color for three fields under the second time setting as described above, from which we could see that the multi-physics contact does not cause any anomalies, as anticipated. Next in Fig. 7, we show the quantitative comparison with the analytical solutions. For the first three sub-figures, the time is fixed at $t = t_{\max}$, so these curves represent the temperature, pore water pressure, and displacement profiles versus depth z . As depicted in (a) and (b), an increase in R_q leads to a postponed heating process and attenuated pressure response. This is because when R_q is larger, the pressure diffusion is faster, so both the temperature and pore water pressure responses become more localized. In (c), we find that the expansion-induced vertical displacement decreases with an increase in R_q due to the aforementioned postponed heating process. Finally, Fig. 7(d) illustrates the response of the effective stress σ'_{xx} with different time intervals when $R_q = 0.5$ (the second time setting). The effective stress σ'_{xx} at the surface is a constant and is always in compression, due to the fixed temperature increase T_1 and constrained lateral expansion. At a depth, pore water pressure has a large effect and the effective stress is in tension. The peak value of σ'_{xx} migrates deeper over time, which corresponds to the migration of the maximum pore water pressure. The numerical results presented in Fig. 7 exhibit a remarkable consistency with analytical solutions, affirming the correctness of the proposed numerical scheme considering multi-physics contact.

4.3. Non-isothermal anisotropic strip load problem

For the last verification example, we consider the traditional strip load problem, but this time we assume the computational domain to be transversely isotropic, and furthermore, we investigate the impacts of higher temperature in the footing section. We will compare the simulation results from SNS-FEM with the standard FEM to ensure our numerical solver is robust under the case of anisotropy.

The geometric configuration and mesh details are depicted in Fig. 8, the finite element mesh consists of 1225 nodes and 2304 elements. In the same figure, we also visualize the inclined bedding orientation at an angle of $\pi/3$ in the Representative Elementary Volume (REV) for

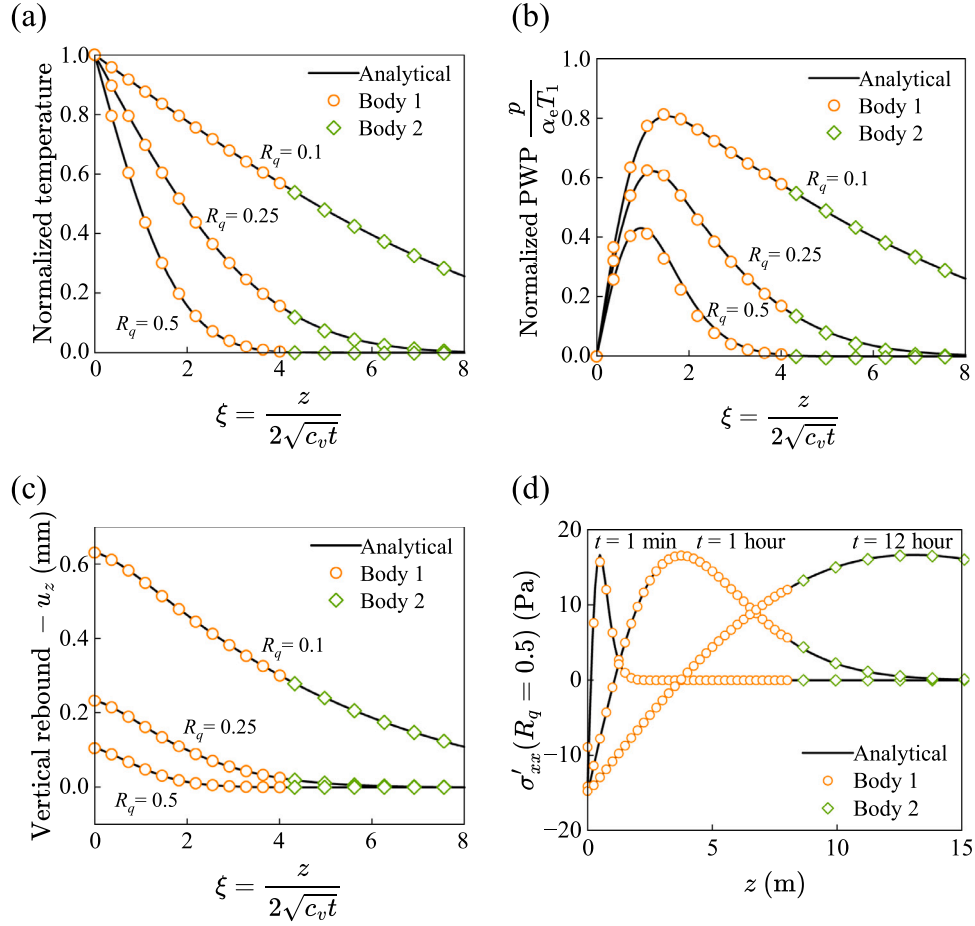


Fig. 7. Comparison with analytical solutions: (a) normalized temperature at $t = t_{\max}$; (b) normalized pore water pressure at $t = t_{\max}$; (c) displacement at $t = t_{\max}$; (d) effective stress σ'_{xx} with $R_q = 0.5$. A good match is achieved between our numerical results (considering two-body contact) and the analytical solutions, and our numerical solver is robust to different time intervals. An increase in R_q leads to a postponed heating process and attenuated pressure response. **For (c), the time level $t = t_{\max}$ must be specified.**

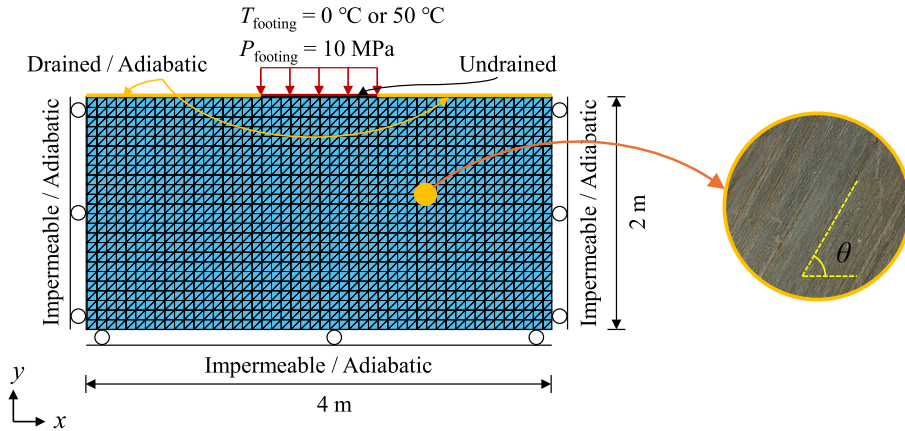


Fig. 8. The geometric configuration and mesh details of the strip load problem. On the right we show the inclined bedding orientation at an angle of $\theta = \pi/3$ in the Representative Elementary Volume (REV) for each macroscopic point.

each macroscopic point. As will be shown later, this inclined bedding would break the symmetry pattern in excess pore water pressure and u_y , despite the hydraulic conductivity tensor is still isotropic. The material parameters are provided in Table 2. For the loading scheme, a uniform strip load of 10 MPa is applied in 0.1 h with $\Delta t = 10^{-3}$ h (100 uniform time steps).

We first conduct the simulation when the temperature at the footing boundary is prescribed at 0°C . Due to a small value of ϖ_θ , this would be equivalent to an isothermal simulation. The comparison results with

standard FEM are given in Fig. 9, from which an excellent consistency is achieved. Due to a small value of the hydraulic conductivity, significant excess pore water pressure is accumulated at the footing boundary, with the maximum value approximately equal to the load magnitude (10 MPa), which is consistent with our intuition. Furthermore, the contour of the excess pore water pressure is skewed due to the strong two-way coupling between the solid deformation and the fluid flow.

Then we increase the prescribed temperature to 50°C at the footing boundary, which in turn has a large impact on both the excess pore

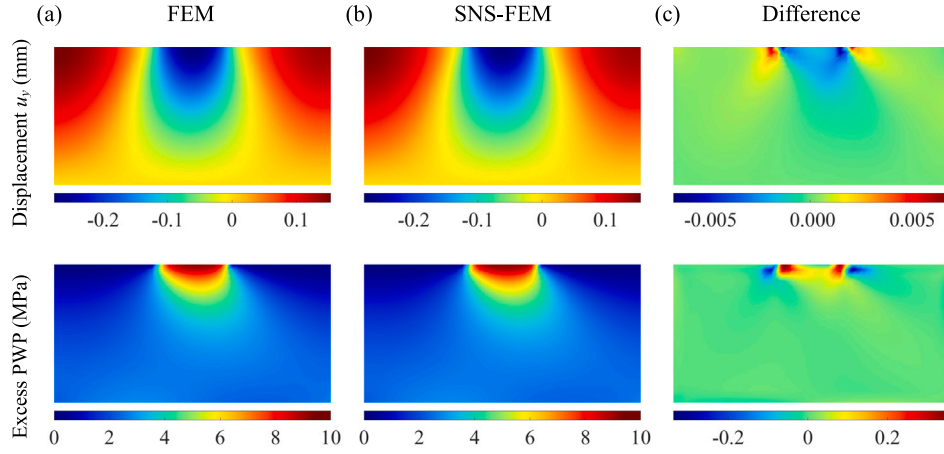


Fig. 9. Comparison of the SNS-FEM with FEM when the temperature at the footing boundary is prescribed at 0 °C. An excellent consistency is achieved between these two methods. Due to an anisotropic $\mathbb{C}^*$ and strong two-way coupling, the contour of the excess pore water pressure is skewed.

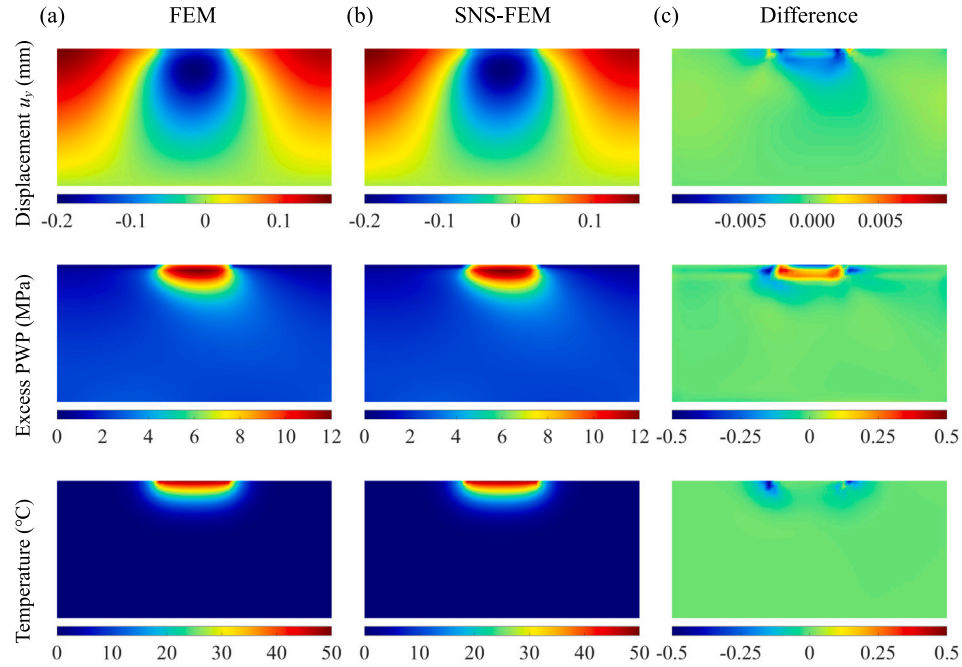


Fig. 10. Comparison of the SNS-FEM with FEM when the temperature at the footing boundary is prescribed at 50 °C. The SNS-FEM still works well in the non-isothermal situation. The temperature field affects the excess pore water pressure field as well as the vertical displacement field.

water pressure and the displacement field, as shown by Fig. 10. This is because the surface heating would generate additional excess pore water pressure, as revealed by the previous example and Fig. 6. In addition, due to the thermal expansion of the footing region, the maximum compression amount in Fig. 10 becomes smaller than that in Fig. 9. Last but not least, in Fig. 11, we depict the results from NS-FEM, i.e., without four virtual integration points, and you could clearly see the results exhibit evident oscillations, which suggests the superiority of SNS-FEM over NS-FEM.

5. Model application

In this part, we provide two application examples, in which the first one considers a hypothetical plane strain compression problem that highlights the role of anisotropy, cross-coupling represented by $\mathcal{K}$, and plastic dissipation; while the second one simulates a more realistic high-temperature pipeline penetration problems.

5.1. Plane strain compression of a transversely isotropic porous medium

If we recall in Eq. (26), we derived a term $\mathcal{K}$ that is related to the cross-coupling, and then in Remark 2.4, we provided a parameter ϖ_θ to measure the intensity of such cross-coupling. For the previous example, the cross-coupling is extremely weak, so therefore, in this application example, we have designed a parameter set such that the cross-coupling cannot be ignored anymore, besides, we have also incorporated the mechanical anisotropy into this example. Gravity is not considered in the simulation, and we ignore all the convection terms in Eqs. (9) and (24).

The example concerns plane strain compression of a transversely isotropic porous medium with dimensions of 0.05 m $\times$ 0.1 m, whose material parameters are provided in Table 3. The computational domain has 561 nodes and 1024 triangle elements. Both the bottom and top edges are clamped, and a uniform compression displacement of 10^{-3} m is applied to the top edge, which is achieved in 5 steps within a time duration of 10 s, i.e., the time step is $\Delta t = 2$ s. The top edge is

Table 2

Material properties of the transversely isotropic porous medium in the strip load problem.

Parameter	Value	Unit
Elasticity [87]:		
λ^e	4270	MPa
a^e	-1870	MPa
b^e	5420	MPa
μ_T^e	9360	MPa
μ_L^e	6510	MPa
Bedding orientation θ	$\pi/3$	rad
Fluid flow:		
k_f	10^{-12}	m/s
γ_f	9.8	kN/m ³
Thermal:		
Θ_0	273.15	K
k_Θ	0.265	kW/m/K
C	2000	kJ/m ³ /K
β_{ave}	1.71×10^{-5}	1/K
α_d	6×10^{-6}	1/K
Other derived quantities:		
Biot tensor b from Eq. (8)	[1, 1, 1, 0]	–
Biot modulus M from Eq. (11)	$+\infty$	GPa
ϖ_Θ from Eq. (30)	3.805×10^{-4}	–

Table 3

Material properties of the transversely isotropic porous medium. According to the value of ϖ_Θ , the effect of $\mathcal{K}$ in Eq. (24) cannot be ignored anymore.

Parameter	Value	Unit
Elasticity [87]:		
λ^e	4270	MPa
a^e	-1870	MPa
b^e	5420	MPa
μ_T^e	9360	MPa
μ_L^e	6510	MPa
K_s	50	GPa
Bedding orientation θ	$\pi/3$	rad
Plasticity [84]:		
c_1^p, c_2^p, c_3^p	0.7, -0.36, 0.6	–
M^p	1.07	–
λ^p	-0.0026	–
$-p_c^p$	40	MPa
Fluid flow:		
k_f	$10^{-13}, 0.1$	m/s
γ_f	9.8	kN/m ³
K_f	2.2	GPa
Thermal:		
Θ_0	273.15	K
k_Θ	10^{-11}	kW/m/K
C	200	kJ/m ³ /K
β_{ave}	1.76522×10^{-3}	1/K
α_d	3×10^{-4}	1/K
Other derived quantities:		
Biot tensor b from Eq. (8)	[0.86, 0.82, 0.8, -0.033]	–
Biot modulus M from Eq. (11)	9664.68	MPa
ϖ_Θ from Eq. (30)	9.5126	–

fully drained for the fluid flow and maintains at 0 °C for the thermal field. For two lateral edges, we have traction-free, zero fluid flux, and adiabatic boundary conditions for displacement, pore water pressure, and temperature fields, respectively.

Fig. 12 depicts the results at the end of the simulation, i.e., $t = 10$ s. As one can see from the panel configuration, we not only investigate the effect of term $\mathcal{K}$, but also consider extremely high and low permeability scenarios. We first consider the extremely low permeability scenario. Without the term $\mathcal{K}$, as shown in the top-left panel (a), the Θ would remain at 0 °C and the THM coupling downgrades to the HM undrained scenario. However, things would change a lot if we consider this term $\mathcal{K}$, as shown in the top-right panel (b). In fact, this cross-coupling term would lead to negative excess pore water pressure and temperature, in other words, $\mathcal{K}$ has a cooling functionality in this THM

undrained and adiabatic scenario. Some quantitative investigations are conducted to explain the appearance of the negative temperature and pore water pressure using the phenomenological porothermoelasticity theory, which is given in Appendix D. Due to the negative temperature and a relatively large α_d , the whole medium would experience a lateral contraction, as shown in the same panel (b).

Now we consider the extremely high permeability scenario. As you can see from the lower half part of Fig. 12, the excess pore water pressure is close to zero, implying a drained condition for the fluid flow. Therefore, the bottom-left panel (c) describes the pure solid deformation. By comparing with the top-left panel (a), we find that the lateral expansion decreases and the u_x contour becomes even more skewed owing to the mechanical anisotropy. For the bottom-right panel (d), since we still have an adiabatic condition for the heat conduction, incorporating the term $\mathcal{K}$ would heat the whole medium because of the heat generated from the deformation. In other words, now $\mathcal{K}$ has a heating functionality. Due to the positive temperature and a relatively large α_d , the whole medium would experience a even larger lateral expansion than the top-left panel (a). Finally, regardless of this term $\mathcal{K}$, the anisotropic elastic stiffness tensor C^e would lead to skewed pressure and temperature contours, even though the fluid flow and thermal parameters are isotropic.

As Fig. 12 only considers anisotropic elasticity, so in our next step, we investigate the temperature increase $\Delta\Theta$ purely due to plastic dissipation $\int \sigma : d\epsilon^p$ under the adiabatic condition. The plasticity model used here is known as the anisotropic modified Cam-Clay (AMCC) model developed for transversely isotropic materials such as shale [96, 112, 113]. We consider two bedding plane orientations $\theta = \pi/4$ and $\theta = \pi/12$, and the loading scheme and boundary conditions are given in Zhang et al. [87]. Results are shown in Fig. 13, which clearly depicts the dependence of plastic dissipation on the bedding plane orientation θ . Furthermore, as we can see from this example, for such a small specific heat value C , which is way less than the real material, the temperature fluctuation $\Delta\Theta$ is still very small. This confirms the conclusion that plastic dissipation is usually not dominant in the temperature evolution equation for soil or rock, so we may not consider it in the linearization process, see Appendix A.

5.2. Pipeline penetration considering thermal effects

The geometry and mesh of the pipeline and seabed soil are illustrated in Fig. 14(a). A non-uniform mesh with 6933 nodes and 13640 triangle elements is used to discretize the seabed soil. The material properties of the soil and pipeline are listed in Table 4. Note for pipeline, we assume a very large Young's modulus of 10^9 kPa and a Poisson's ratio of 0.3. The initial temperature is the same as the reference temperature.

The pipeline velocity during penetration is maintained at a constant of 1 m/day, similar to the study by Carbonell et al. [14], and the time increment per step is set to $\Delta t = 0.01$ day. The total number of time steps is 100. The boundary conditions necessitate further consideration, accounting for the THM coupled interactions between the soil and pipeline. Upon contact with the pipeline, the seabed soil promptly attains the surface temperature of the pipeline in the corresponding portion. Additionally, that portion changes to an undrained state upon contact with the pipeline, and upon disengagement, it returns to a fully drained and adiabatic state.³ Besides, it is observed that the

³ This assumption of the adiabatic boundary condition is quite strong. Meanwhile, we realize that this model is still idealized compared to reality. We have not considered the effect of seawater above the soil body, and we assume that heat exchange between the saturated soil and the surrounding air is extremely slow, whereas the heat exchange between the soil and the pipeline is the opposite. Therefore, the top boundary of the soil body is set as an adiabatic boundary. From the following results, we can observe that the heat conduction process is quite slow. Thus, we believe that applying Dirichlet boundary conditions on boundaries that are far from the pipeline should also be effective and would yield the same results.

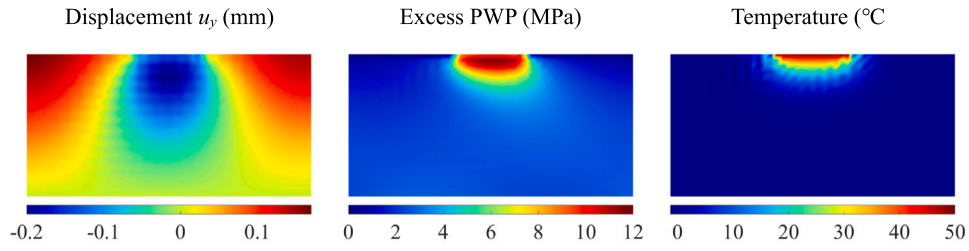


Fig. 11. Simulation results from NS-FEM without four virtual integration points for the same problem. All the field results exhibit evident oscillations, which suggests the superiority of SNS-FEM over NS-FEM.

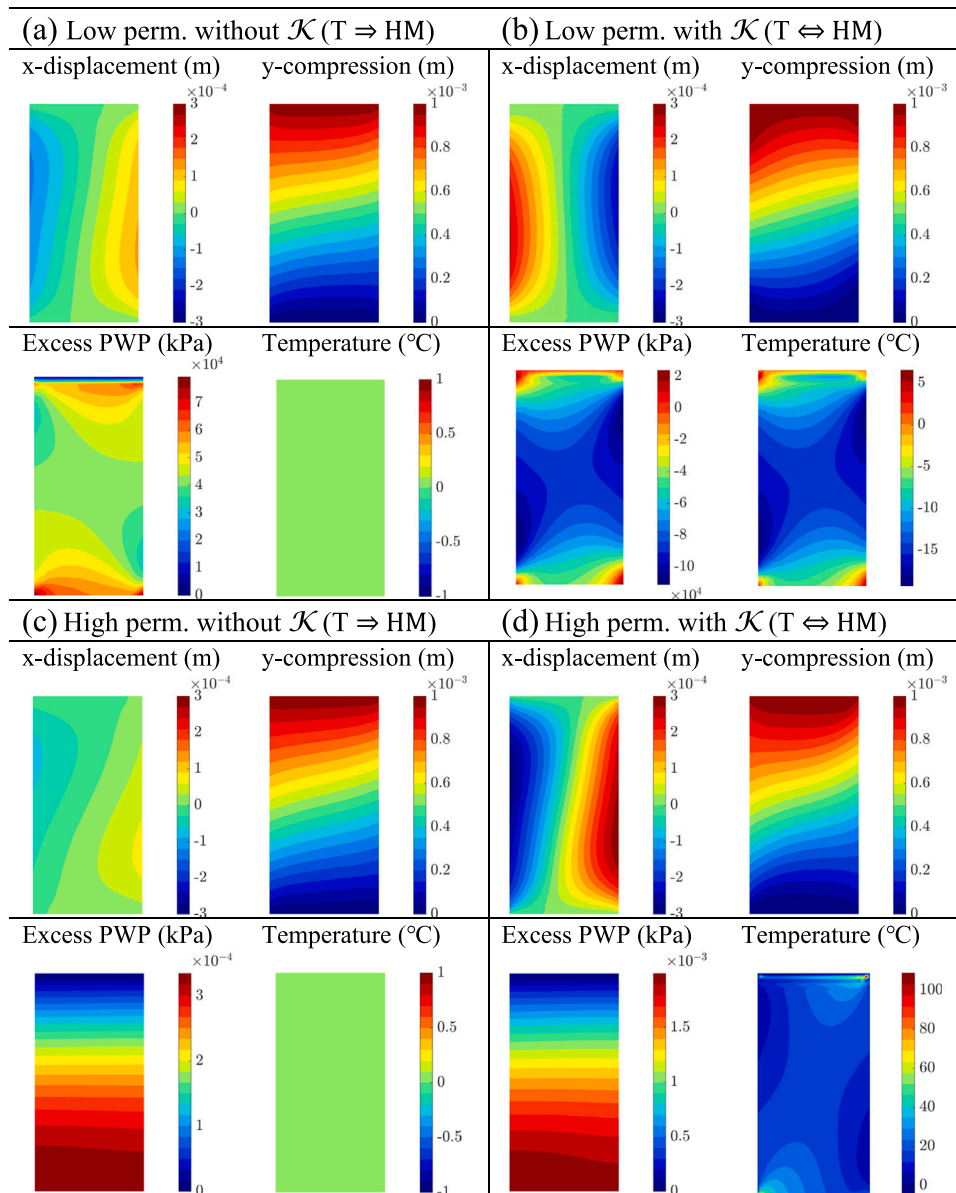


Fig. 12. Impacts of permeability, mechanical anisotropy, and cross-coupling term $\mathcal{K}$ on the undrained/drained and adiabatic responses of the porous medium. The term $\mathcal{K}$ could have either heating or cooling functionality in TM or THM scenario. The anisotropy in $\mathbb{C}^*$ would affect the other two fields through the strong two-way coupling.

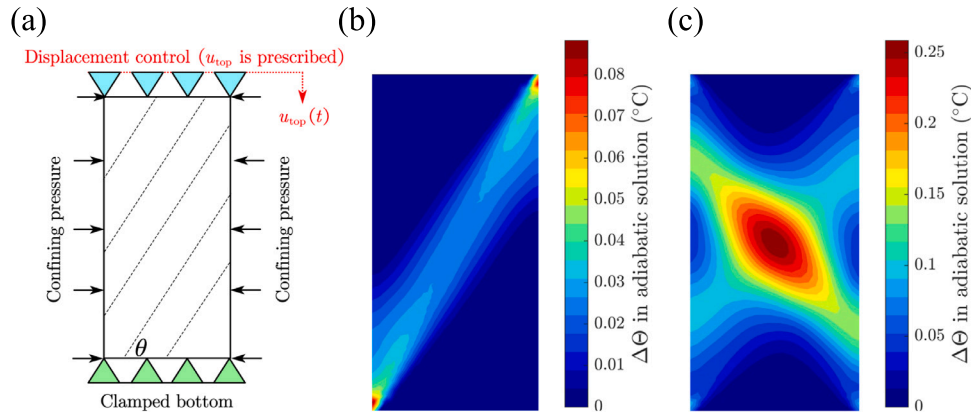


Fig. 13. Plane strain compression of a transversely isotropic shale specimen with clamped end: (a) schematic diagram; contours of temperature increment for (b) bedding plane orientation $\theta = \pi/4$ and (c) bedding plane orientation $\theta = \pi/12$.

Table 4

Properties of the seabed soil and pipeline. According to the value of ϖ_θ , the thermally uncoupled form can be used here. Gravity is explicitly neglected in the calculation.

Parameter	Value	Unit
Reference temperature θ_0	278.15	K
Pipeline temperature T_{pipe}	5, 100, 195	°C
Contact friction coefficient μ	0, 0.3	–
Thermal conductivity coefficient k_θ	2.5	W/m/K
Overall specific heat per unit volume C	2636	kJ/m ³ /K
Fluid-specific heat capacity c_f	4180	J/kg/K
Thermal expansion coefficient β_{ave}	8.4×10^{-5}	1/K
β_f from Eq. (4) for fluid	3×10^{-4}	1/K
Linear thermal strain coefficient α_d	10^{-5}	1/K
Young's modulus E	2	MPa
Poisson's ratio ν	0.3	–
Cohesion	10	kPa
Friction angle φ	0, 20	deg
Dilatancy angle ψ	0, 0.5	deg
Porosity ϕ	0.2	–
Biot coefficient	1	–
Fluid bulk modulus K_f	2.2	GPa
Fluid specific weight γ_f	9.8	kN/m ³
Hydraulic conductivity coefficient k_f	10^{-8}	m/s
Dimensionless coupling parameter ϖ_θ	1.55×10^{-7}	–

issue of pipeline penetration typically necessitates remeshing, a process of redefining the computational grid to accommodate the changing geometry. Details can be found in Fang et al. [107].

We begin by conducting a pure solid deformation analysis and comparing the normalized resistance result with that of Zhou et al. [114]. The entire model's dimensions are reduced to 1/25 of their original sizes for comparison. The cohesion is changed to 5 kPa (Tresca plasticity model) and we assume frictionless contact. Other parameters remain the same with Table 4. As shown in Fig. 14(b), the vertical resistance applied to the pipeline during penetration increases with a decreasing slope. When the pipeline starts to buckle laterally, the vertical resistance experiences a sudden drop, and the horizontal resistance undergoes a sharp increase. The results obtained by SNS-FEM matches well with the reference results and are even more smooth. We also perform a qualitative comparison with the experimental findings regarding the soil deformed shape during the pipeline penetration. We could find that for either large or small Young's modulus E , the soil deformed shape generally agrees with that from the centrifuge experiment [10] or model test, as shown in Fig. 14(c).

Now we consider the multi-physics coupling and display the temporal evolution of important quantities during the pipeline penetration process, as shown in Fig. 15. At the shallow penetration depth, we could clearly see the logspiral shear failure zone extended from both sides of the pipeline, and it extends until the top surface. As a result, we can see the local soil uplift phenomenon. For excess pore water pressure

(PWP), it has a distinct pattern that cannot be easily predicted using one's intuition. The maximum PWP occurs right below the pipeline, but at a certain distance from it. Negative PWP happens at two lateral sides of the pipeline. Lastly, for the temperature field, the high temperature only happens in a tiny region adjacent to the contact surface, due to the slow heat conduction process.

Next, we look at some quantitative results on the penetration resistance F_y assuming a unit thickness, as shown in Fig. 16. In the numerical experiment design, we consider both frictional contact with friction coefficient $\mu = 0.3$ and frictionless contact with $\mu = 0$; three pipeline temperatures: $T_{\text{pipe}} = 5, 100, 195$ °C; Tresca plasticity model ($\varphi = \psi = 0$) and Mohr–Coulomb plasticity model ($\varphi = 20^\circ, \psi = 0.5^\circ$); thermo-hydro-mechanical (THM) coupling and thermo-mechanical (TM) coupling, which leads to a total of 24 cases.

Firstly, it is evident that the frictional contact scenario consistently predicts a higher penetration resistance [14]. This is attributable to the additional frictional force acting along the contact surface. Secondly, when comparing the resistance outcomes between THM coupling and TM coupling, we observe that the TM coupling yields a higher penetration resistance than THM coupling for the Mohr–Coulomb model, see Fig. 16(c) and (d), while for the Tresca model, TM and THM coupling yield close resistance values, see Fig. 16(a) and (b). This is because the Mohr–Coulomb yield criterion is pressure-dependent. It also suggests that the use of Tresca model in the hydro-mechanical coupling analysis is not recommended as the total stress analysis would yield similar results and is more computationally efficient. Thirdly, due to small values of α_d , in TM coupling, the resistance is insensitive to the pipeline temperature T_{pipe} , as shown by Fig. 16(b) and (d). Finally, in THM coupling, the effect of T_{pipe} on penetration resistance varies between the Tresca and Mohr–Coulomb models. In the Tresca model as shown by Fig. 16(a), resistance is insensitive to T_{pipe} when $\mu = 0$, but this is not the case for the Mohr–Coulomb model, see Fig. 16(c). We think it could be attributed to the fact that, for the Mohr–Coulomb model, the increase in excess pore water pressure (due to the increase of T_{pipe}) is attenuated, compared to the Tresca model, as verified in Fig. 17(c) and (d). Nevertheless, for the frictional contact, lower resistance is observed at higher T_{pipe} for both constitutive models, as verified in Fig. 16(a) and (c).

6. Closure

We have presented a continuum framework for thermo-hydro-mechanical coupling in anisotropic elastoplastic porous media, an extension of the work done by Zhao and Borja [84] for isothermal porous media. Therefore, besides the two effective stress measures and an alternative strain tensor field that emanated naturally from the formulation, the derivation also resulted in a generalized bulk average volumetric thermal expansion coefficient β_{ave} , which is a function of β_s

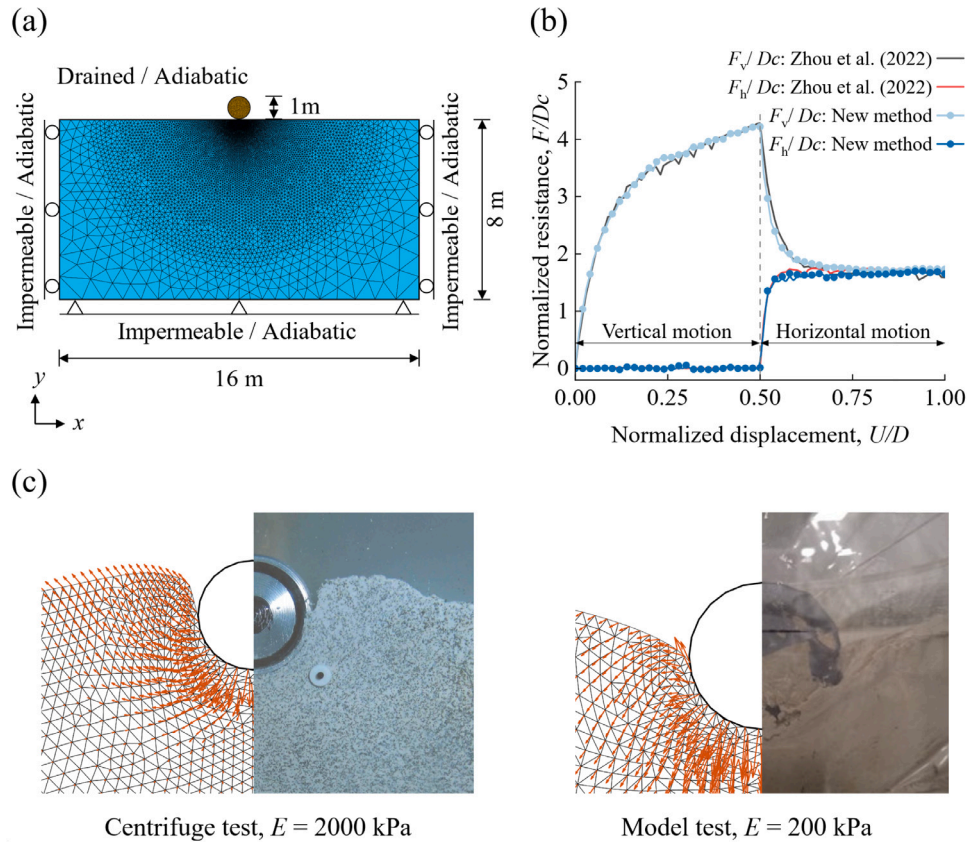


Fig. 14. Pipeline penetration: (a) geometry and mesh of the pipeline and seabed soil; (b) quantitative comparison with the reference work [114] in terms of the normalized penetration resistance (unit thickness) where D is the pipeline diameter; (c) qualitative comparison with the centrifuge experiment [10] or model test in terms of the soil deformed shape.

for solid constituent, β_f for fluid constituent, α_d for solid skeleton, and anisotropic Biot tensor $\mathbf{b}$. In particular, α_d could be anisotropic as well. It is the first time, to the best of the authors' knowledge, that such a comprehensive formula for β_{ave} is derived.

The second interesting finding resulted from the formulation is an analogous poro-elasto-plastic structural heating/cooling term $\mathcal{K}$ that bridges not only to the phenomenological porothermoelasticity but also to the uncoupled standard thermal equation. The relative importance of this term $\mathcal{K}$ is represented by a dimensionless number ϖ_θ and we have revised its definition to consider anisotropy. In this work, the stabilized node-based smoothed finite element method is adopted as the main numerical tool. Numerical simulation reveals that when ϖ_θ is on the order of 1, $\mathcal{K}$ could have the cooling functionality in the THM coupled scenario, and have the heating functionality in the TM coupled scenario. The anisotropy in the elastic tensor $\mathbb{C}^e$ would affect the other two fields through the strong two-way coupling. We also quantify the temperature fluctuation purely from plastic dissipation through the plane strain compression of a transversely isotropic shale specimen using the AMCC model. Finally, more realistic pipeline penetration simulations are carried out to highlight the impacts of pipeline temperature, contact friction coefficient, constitutive model, and multi-physics coupling on the ensuing mechanical responses such as the penetration resistance.

Overall, this study offers a revisit to the THM coupling theory with promising findings, an important extension of the smoothed finite element method to THM coupling, and a comprehensive understanding of complex geomechanics problems.

CRediT authorship contribution statement

Xian-Han Wu: Writing – original draft, Validation, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Qi Zhang:** Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization. **Wei-Qiang Feng:** Writing – review & editing, Supervision, Resources, Project administration, Investigation, Funding acquisition, Formal analysis. **Zhen-Yu Yin:** Writing – review & editing, Supervision, Resources, Project administration, Methodology, Funding acquisition, Conceptualization. **Huangcheng Fang:** Writing – review & editing, Software, Methodology, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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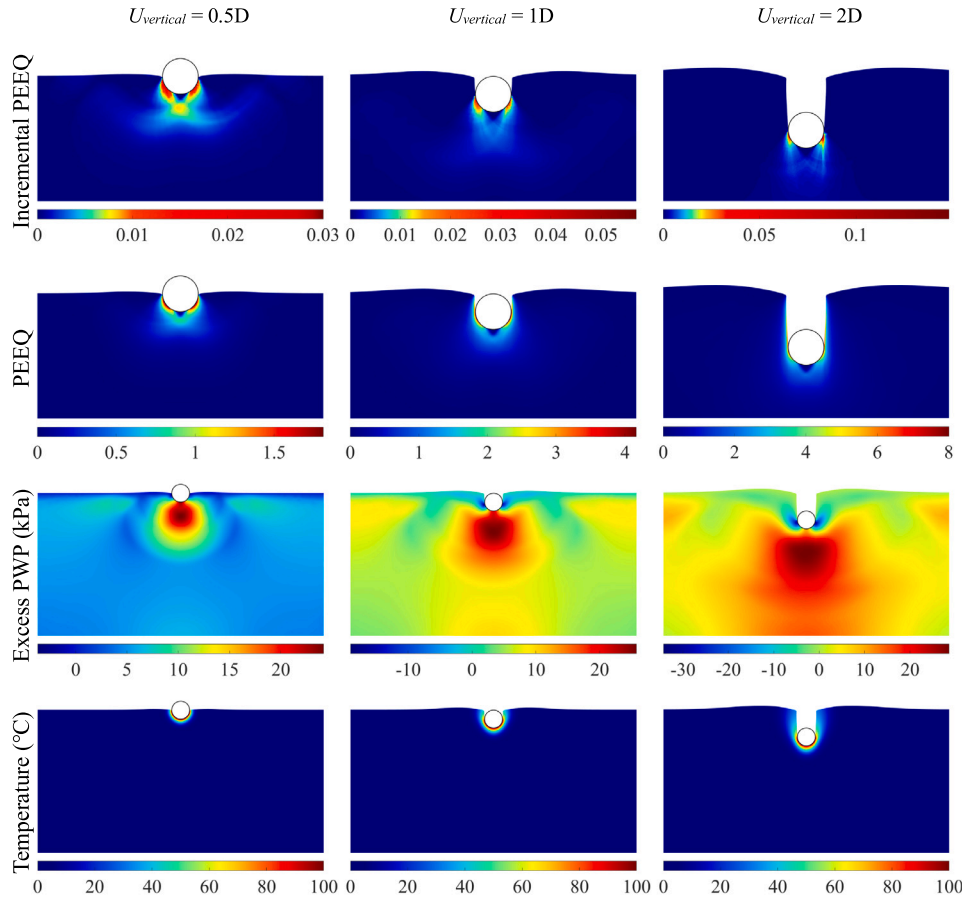


Fig. 15. Equivalent plastic strain (plotted in a zoomed-in area surrounding the pipeline), excess pore water pressure, and temperature distributions during the pipeline penetration process ($\mu = 0.3$, $T_{\text{pipe}} = 100^\circ\text{C}$, and Mohr–Coulomb plasticity model is adopted). The incremental PEEQ represents the equivalent plastic strain increment within a time step.

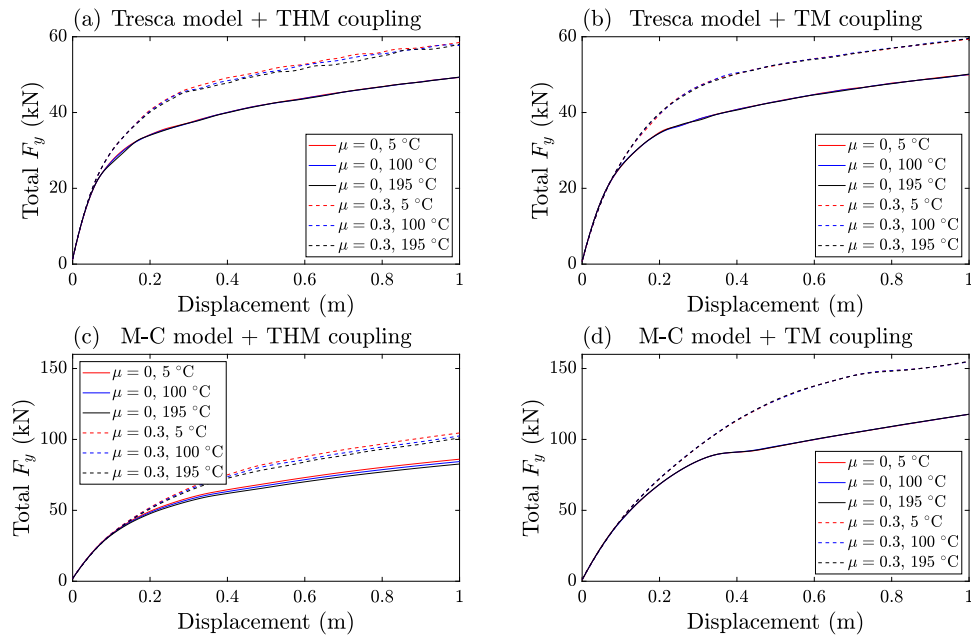


Fig. 16. Effect of friction coefficient μ , pipeline temperature T_{pipe} , constitutive model, and multi-physics coupling on the total penetration resistance F_y . Frictional contact scenario consistently predicts a higher penetration resistance, and it makes little difference between TM and THM coupling when you adopt the Tresca constitutive model. Therefore, use of Tresca model in the hydro-mechanical coupling analysis is not recommended.

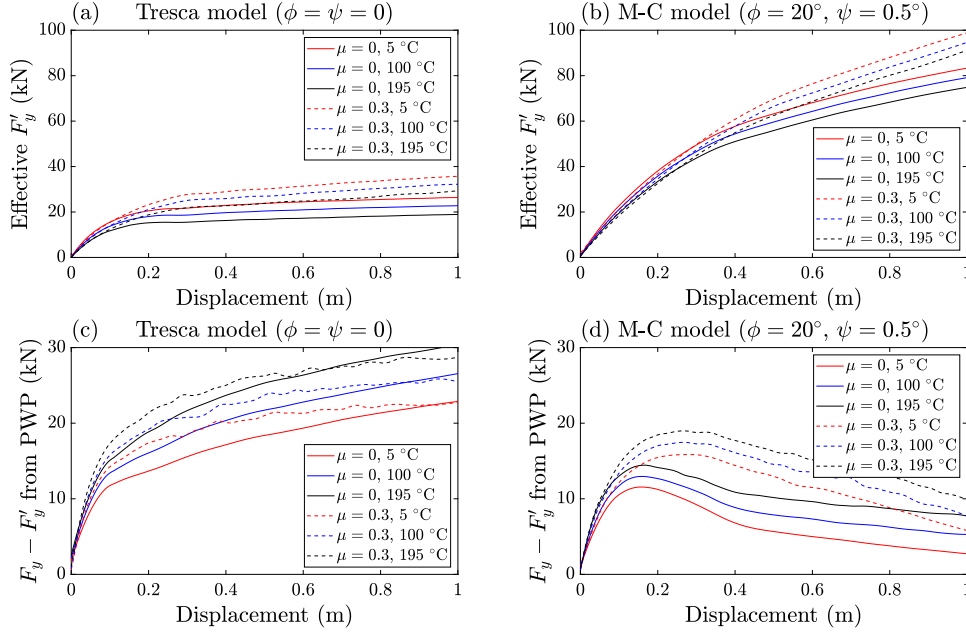


Fig. 17. Decomposition of the total penetration resistance F_y into two parts, i.e., the contribution from the effective stress F'_y , as shown in (a) and (b), and from the pore water pressure (PWP), as shown in (c) and (d). The effect of T_{pipe} on penetration resistance varies between the Tresca and Mohr-Coulomb constitutive models.

Appendix A. Linearization of the residual equation

The relations below are used in the linearization of the residual equation

$$\frac{\partial \boldsymbol{\varepsilon}^e}{\partial \boldsymbol{\varepsilon}} = \mathbb{S}^e : \mathbb{C}^{ag}, \quad (\text{A.1})$$

$$\frac{\partial \boldsymbol{\varepsilon}^e}{\partial p} = \frac{\mathbb{S}^e : \mathbb{C}^{ag} : \mathbf{1} - \mathbf{1}}{3K_s}, \quad (\text{A.2})$$

$$\frac{\partial \boldsymbol{\varepsilon}^e}{\partial \boldsymbol{\theta}} = -\mathbb{S}^e : \mathbb{C}^{ag} : \boldsymbol{\alpha}_d, \quad (\text{A.3})$$

where $\mathbb{S}^e = (\mathbb{C}^e)^{-1}$ is the fourth-order elastic compliance tensor, and $\mathbb{C}^{ag}$ is the fourth-order algorithmic elastoplastic tangent modulus tensor. Note the plasticity model for the solid skeleton is formulated in terms of the Terzaghi effective stress and an alternative strain tensor field $\tilde{\boldsymbol{\varepsilon}}$

$$\tilde{\boldsymbol{\varepsilon}} = (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\theta) + \frac{p\mathbf{1}}{3K_s}. \quad (\text{A.4})$$

As a result, the submatrices in the algorithmic tangent operator $\mathcal{K}$ derived from linearization take the following forms

$$\mathcal{K}_{uu} = \int_B \mathbb{B}^T \{ \mathbb{C}^{ag} \} \mathbb{B} dV, \quad (\text{A.5})$$

$$\mathcal{K}_{up} = - \int_B \mathbb{B}^T \{ \mathbb{b}^{eg} \} \mathbb{N}_p dV, \quad (\text{A.6})$$

$$\mathcal{K}_{u\theta} = - \int_B \mathbb{B}^T \{ \mathbb{C}^{ag} : \boldsymbol{\alpha}_d \} \mathbb{N}_\theta dV, \quad (\text{A.7})$$

$$\mathcal{K}_{pu} = \int_B \mathbb{N}_p^T \{ \hat{\mathbf{b}}^{ep} \} \mathbb{B} dV, \quad (\text{A.8})$$

$$\begin{aligned} \mathcal{K}_{pp} = & \int_B \frac{1}{M} \mathbb{N}_p^T \mathbb{N}_p dV + \Delta t \int_B \mathbb{E}_p^T \frac{\mathbf{k}_f}{\gamma_f} \mathbb{E}_p dV \\ & + \Delta t \int_B \mathbb{N}_p^T \frac{\mathbf{q}_f}{K_f} \mathbb{E}_p dV - \Delta t \int_B \mathbb{N}_p^T \left(\frac{\nabla \rho_f}{\rho_f} \cdot \frac{\mathbf{k}_f}{\gamma_f} \right) \mathbb{E}_p dV \\ & + \int_B \mathbb{N}_p^T \frac{\mathbf{1} : (\mathbb{C}^e - \mathbb{C}^{ag}) : \mathbf{1}}{9K_s^2} \mathbb{N}_p dV, \end{aligned} \quad (\text{A.9})$$

$$\mathcal{K}_{p\theta} = - \int_B \hat{\rho}_{\text{ave}}^{ep} \mathbb{N}_p^T \mathbb{N}_\theta dV - \Delta t \int_B \beta_f \mathbb{N}_p^T \mathbf{q}_f \mathbb{E}_\theta dV, \quad (\text{A.10})$$

$$\mathcal{K}_{\theta u} = \int_B \boldsymbol{\theta}_0 \mathbb{N}_\theta^T \{ \boldsymbol{\alpha}_d : \mathbb{C}^{ag} \} \mathbb{B} dV, \quad (\text{A.11})$$

$$\mathcal{K}_{\theta p} = - \int_B \hat{\rho}_{\text{ave}}^{ep} \boldsymbol{\theta}_0 \mathbb{N}_\theta^T \mathbb{N}_p dV - \Delta t \int_B \rho_f c_f \mathbb{N}_\theta^T \nabla \boldsymbol{\theta} \cdot \frac{\mathbf{k}_f}{\gamma_f} \mathbb{E}_p dV, \quad (\text{A.12})$$

$$\begin{aligned} \mathcal{K}_{\theta\theta} = & \int_B \mathbb{C} \mathbb{N}_\theta^T \mathbb{N}_\theta dV + \Delta t \int_B \mathbb{E}_\theta^T \mathbf{k}_\theta \mathbb{E}_\theta dV \\ & + \Delta t \int_B \rho_f c_f \mathbb{N}_\theta^T \mathbf{q}_f \mathbb{E}_\theta dV \\ & + \int_B \boldsymbol{\theta}_0 \mathbb{N}_\theta^T [\boldsymbol{\alpha}_d : (\mathbb{C}^e - \mathbb{C}^{ag}) : \boldsymbol{\alpha}_d] \mathbb{N}_\theta dV, \end{aligned} \quad (\text{A.13})$$

where the curly bracket $\{ \circ \}$ implies the Voigt notation, and for the sake of conciseness, we define two analogous elastoplastic Biot tensors as well as two elastoplastic thermal expansion coefficients as

$$\mathbf{b}^{ep} = \mathbf{1} - \frac{\mathbb{C}^{ag} : \mathbf{1}}{3K_s}; \quad \hat{\mathbf{b}}^{ep} = \mathbf{1} - \frac{\mathbf{1} : \mathbb{C}^{ag}}{3K_s}; \quad (\text{A.14})$$

$$\beta_{\text{ave}}^{ep} = \beta - \boldsymbol{\alpha}_d : (\mathbf{1} - \mathbf{b}^{ep}); \quad \hat{\beta}_{\text{ave}}^{ep} = \beta - (\mathbf{1} - \hat{\mathbf{b}}^{ep}) : \boldsymbol{\alpha}_d.$$

Some important observations are summarized here in bullet form.

- The reason that we have $\mathbf{b}^{ep}$ and $\hat{\mathbf{b}}^{ep}$ simultaneously is because $\mathbb{C}^{ag}$ may not satisfy the major symmetry property, owing to the non-associative flow rule or the incremental hardening law [96].
- When $\mathbf{b} = \mathbf{1}$, K_s becomes infinite, and as a result, $\mathbf{b}^{ep} = \hat{\mathbf{b}}^{ep} = \mathbf{1}$ and $\beta_{\text{ave}}^{ep} = \hat{\beta}_{\text{ave}}^{ep} = \beta$, the contributions of plasticity modeling reflected with $\mathbb{C}^{ag}$ only enter the temperature and deformation equations. This is consistent with our expectation.
- When $\mathbb{C}^{ag} = \mathbb{C}^e$, i.e., elastic deformation, $\mathbf{b}^{ep}$ and $\hat{\mathbf{b}}^{ep}$ automatically change to $\mathbf{b}$, and β_{ave}^{ep} and $\hat{\beta}_{\text{ave}}^{ep}$ change to β_{ave} . The last term in $\mathcal{K}_{pp}$ and $\mathcal{K}_{\theta\theta}$ also drops out.
- The linearization of $\mathcal{D}_{\text{mech}}^p$ and $\mathcal{D}_{\text{hydro}}$ is ignored in $\mathcal{K}$, assuming their contribution to the temperature fluctuation is small, see Fig. 13.

Table B.5

Time step settings tested by Cui et al. [95].

Scheme 1		Scheme 2	
Step	Time step size (s)	Step	Time step size (s)
1–10	0.01	1–10	1
11–20	0.1	11–10010	10
21–30	10		
31–40	100		
41–140	1000		

In the above submatrices, some field quantities are not written explicitly in terms of the shape functions and its derivatives, which is supplemented here

$$q_f = -\frac{k_f}{\gamma_f} \cdot (\mathbb{E}_p p - \rho_f g); \quad \frac{\nabla \rho_f}{\rho_f} = \frac{\mathbb{E}_p p}{K_f} - \beta_f \mathbb{E}_\Theta \Theta; \quad \nabla \Theta = \mathbb{E}_\Theta \Theta, \quad (\text{A.15})$$

where p and Θ are the nodal column vectors for pore water pressure and temperature fields, respectively.

Appendix B. Convergence and sensitivity analysis of one-dimensional thermo-elastic consolidation

The one-dimensional thermo-elastic consolidation analysis of a fully saturated column of soil was firstly presented by Aboustit et al. [23] and has since served as a benchmark for thermo-hydro-mechanical (THM) formulations. In this analysis, the upper boundary of the soil column is fully drained, whereas the remaining boundaries are impermeable, thermally isolated, and constrained in the normal direction. The properties of the soil column are almost the same as Table 1, expect the Young's modulus E is decreased to 6 kPa (therefore the α_0 is even smaller) and hydraulic conductivity k_f is 3.92×10^{-2} m/s. β_i is also changed to $9E-7$ 1/K, but we forgot to mention! Thus β_{ave} is also changed!

Here, we take the results from Cui et al. [95] as the reference. Table B.5 shows the two types of time step settings tested by Cui et al. [95]. Scheme 2 was found to better capture the maximum settlement compared to Scheme 1. However, now we find that simulation results using a uniform time step at the logarithmic scale can be as good as, if not better than, Scheme 2 from Cui et al. [95], as shown by Fig. B.18(b) below. In fact, the simulation using 200 steps could achieve a fair match with Scheme 2, which has 10 010 steps. This is a huge improvement.

Since this example only has numerical solutions and lacks analytical solutions, we analyze the mesh size and time step sensitivity from the perspective of maximum settlement. Fig. B.18(c) demonstrates that the maximum settlement gradually stabilizes as the mesh size decreases. Fig. B.18(d) indicates that the maximum settlement also tends to converge as the number of time steps increases.

Appendix C. Summary of the analytical solutions for heating of a half space

We first provide the formula for R_q

$$R_q = \sqrt{\frac{c_v}{\kappa_\Theta}}, \quad (\text{C.1})$$

where $c_v = k_f E_{1d} / \gamma_f$ is the consolidation coefficient, $\kappa_\Theta = k_\Theta / C$ is the heat diffusivity coefficient, and $E_{1d} = E(1-\nu)/(1+\nu)/(1-2\nu)$ is the one-dimensional constrained modulus. Then we define the dimensionless coordinate ξ as

$$\xi = \frac{z}{2\sqrt{c_v t}}. \quad (\text{C.2})$$

Finally, the analytical solutions for temperature, pore water pressure, vertical displacement, and horizontal effective stress are now expressed in terms of ξ as [38]

$$\frac{\Theta(\xi)}{T_1} = \text{erfc}(R_q \xi), \quad (\text{C.3})$$

$$\frac{p(\xi)}{\alpha_e T_1} = \frac{1}{1-R_q^2} [\text{erfc}(R_q \xi) - \text{erfc}(\xi)], \quad (\text{C.4})$$

$$u_z(\xi) = -\frac{2T_1\sqrt{c_v t}}{E_{1d}} \left\{ \frac{E\alpha_d}{1-2\nu} \left[\frac{e^{-R_q^2 \xi^2}}{\sqrt{\pi} R_q} - \xi \text{erfc}(R_q \xi) \right] + \frac{\alpha_e}{1-R_q^2} \left[\xi \text{erfc}(\xi) - \xi \text{erfc}(R_q \xi) + \frac{e^{-R_q^2 \xi^2}}{\sqrt{\pi} R_q} - \frac{e^{-\xi^2}}{\sqrt{\pi}} \right] \right\}, \quad (\text{C.5})$$

$$\sigma'_{xx}(\xi) = \frac{\nu}{1-\nu} p - \frac{E\alpha_d}{1-\nu} \Theta, \quad (\text{C.6})$$

where $\text{erfc}(x)$ is the complementary error function, and α_e is an intermediate parameter given as

$$\alpha_e = E_{1d} \beta_{ave} - \frac{E\alpha_d}{1-2\nu}. \quad (\text{C.7})$$

Please note that the above analytical expressions are thermally uncoupled solutions and do not consider convection [38], which means that they are only valid when $\varpi_\Theta \ll 1$.

Another subtle point is that $p(\xi)$ and $u_z(\xi)$ are not well-defined for the case of $R_q = 1$, and in that case, we need to take the limit using the L'Hôpital's rule, and the result becomes

$$\frac{p(\xi)}{\alpha_e T_1} = \frac{1}{\sqrt{\pi}} \xi e^{-\xi^2}, \quad (\text{C.8})$$

$$u_z(\xi) = -\frac{2T_1\sqrt{c_v t}}{E_{1d}} \left\{ \frac{E\alpha_d}{1-2\nu} \left[\frac{e^{-\xi^2}}{\sqrt{\pi}} - \xi \text{erfc}(\xi) \right] + \frac{\alpha_e}{2\sqrt{\pi}} e^{-\xi^2} \right\}. \quad (\text{C.9})$$

Appendix D. Investigation of the negative temperature and pore water pressure in Fig. 12 (b)

Phenomenological porothermoelasticity derived in Cheng [38] provided the following “constitutive” equations for the fluid content variation ζ and total entropy density s as

$$\zeta = \hat{\alpha} \varepsilon_v + \frac{p}{\hat{M}} - \hat{\beta}_e \Theta, \quad (\text{D.1})$$

$$s = \hat{\alpha}_d \varepsilon_v - \hat{\beta}_e p + \frac{\hat{c}_d}{\Theta_0} \Theta, \quad (\text{D.2})$$

where we add the $\hat{(\cdot)}$ symbol on top of the material parameters to emphasize the different notations used in the current work and that of Cheng [38]. Note that the above two equations Eqs. (D.1) and (D.2) are only valid for the case of isotropy. However, as we are not interested in the exact values of p and Θ , what matters the most are their signs and relative magnitude. The isotropic form is sufficient.

Therefore, the next step is to convert our anisotropic material parameters to these approximate isotropic material parameters. In this work, the $\hat{\alpha}$ and $\hat{\alpha}_d$ are obtained by taking the one-third of the trace of their anisotropic counterparts, and the calculation is shown below

$$\hat{\alpha} = \frac{1}{3} \text{Tr}(\hat{\alpha}) = \frac{1}{3} \text{Tr} \begin{pmatrix} 0.86 & -0.033 & 0 \\ -0.033 & 0.82 & 0 \\ 0 & 0 & 0.8 \end{pmatrix} = 0.828, \quad (\text{D.3})$$

$$\hat{\alpha}_d = \frac{1}{3} \text{Tr}(\hat{C}^e : \alpha_d) = \frac{1}{3} \text{Tr} \begin{pmatrix} 6.29 & 1.51 & 0 \\ 1.51 & 8.03 & 0 \\ 0 & 0 & 8.9 \end{pmatrix} = 7.74 \text{ [MPa/K]}. \quad (\text{D.4})$$

For other parameters in Eqs. (D.1) and (D.2), we can get their values from Table 3 by identifying the equivalence between different notations, and the results are: $\hat{M} = M = 9664.68$ MPa, $\hat{\beta}_e = \beta_{ave} = 1.76522 \times 10^{-3}$ 1/K, and $\hat{c}_d = C = 200$ kJ/m³/K.

In the scenario of Fig. 12(b), both the thermal and hydraulic conductivities are extremely small, and therefore, the changes of ζ and s are almost zero. Therefore, we could have the following two equations with two unknowns as

$$\begin{cases} \Delta p / \hat{M} - \hat{\beta}_e \Delta \Theta = -\hat{\alpha} \Delta \varepsilon_v \\ -\hat{\beta}_e \Delta p + (\hat{c}_d / \Theta_0) \Delta \Theta = -\hat{\alpha}_d \Delta \varepsilon_v \end{cases}. \quad (\text{D.5})$$

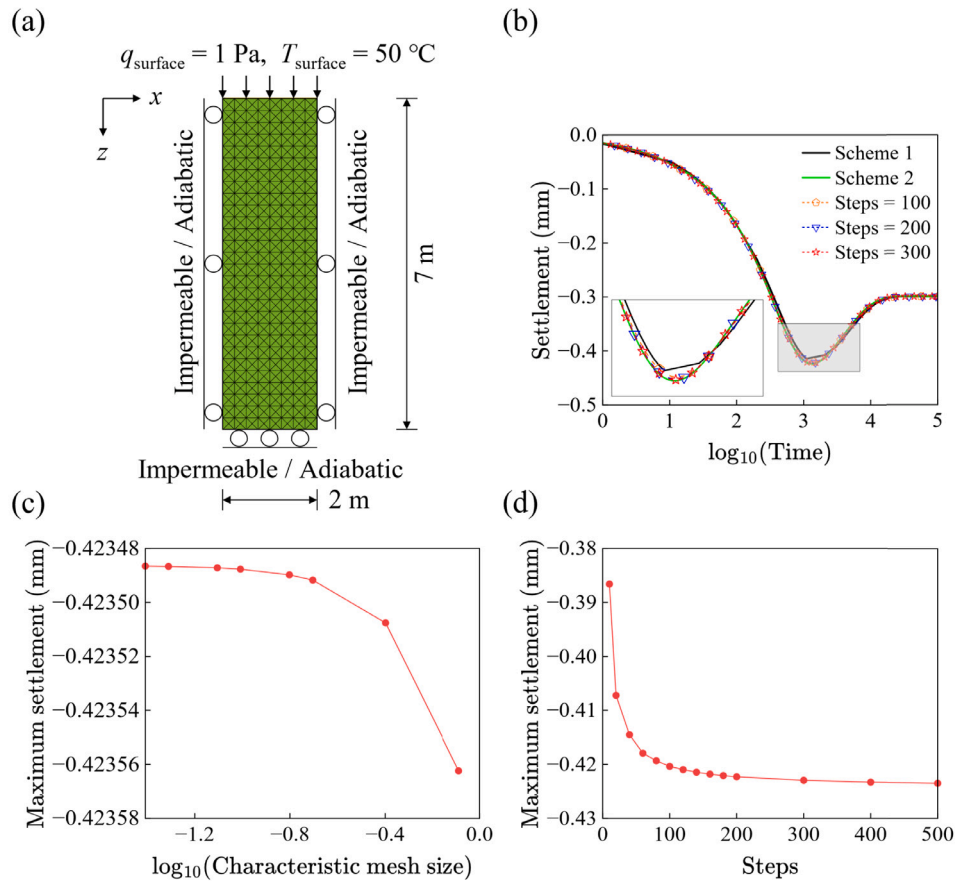


Fig. B.18. Thermal consolidation example: (a) geometry and mesh of the soil column with 485 nodes and 896 elements; (b) settlement curves of different time step settings; (c) maximum settlement versus different characteristic mesh size while the total number of time steps is fixed at 500 (uniform time step at the logarithmic scale); (d) maximum settlement versus different total number of time steps (uniform time step at the logarithmic scale, mesh size is adopted from (a)).

Again, since we are only interested in the sign of p and θ and their relative magnitude, without loss of generality, we could assign $\Delta\epsilon_v = -0.01$ and solve Eq. (D.5). Here the sign of $\Delta\epsilon_v$ is important, because the whole specimen is in the compression state, as shown in Fig. 12(b), we need to set $\Delta\epsilon_v$ to be negative. The solutions to Eq. (D.5) are $\Delta p = -46.9 \text{ MPa}$ and $\Delta\theta = -7.44 \text{ }^{\circ}\text{C}$, and their relative magnitude ratio is about 6.3 MPa/ $^{\circ}\text{C}$, which approximately matches with the colorbar ticks of p and θ in Fig. 12(b).

Note for Fig. 12(d), only the second sub-equation of Eq. (D.5) holds and $\Delta p \approx 0$, so we know $\Delta\theta$ is positive when $\Delta\epsilon_v < 0$. To summarize, by using the phenomenological porothermoelasticity theory, we have successfully justified the appearance of negative temperature and pore water pressure in Fig. 12(b) and positive temperature in Fig. 12(d).

Appendix E. Supplementary data

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.jimecs.2025.110087>.

Data availability

Data will be made available on request.

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